PTOLEMY

DAY 1

INTRODUCTION TO PTOLEMY AND HIS *ALMAGEST* AND FIRST BACK YARD EXERCISES

INTRODUCTION.

Little is known about Claudius Ptolemaeus. He was probably born at Ptolemais Hermii, and lived from around 100 A.D. to around 178, which means that all his astronomy was obviously "naked eye" astronomy, unassisted by binoculars or telescopes. He did use and in some cases develop certain instruments (e.g. the astrolabe) for accurately measuring angular "distances" between celestial objects—for example, the angle formed by any two stars with the observer's eye at the vertex. Even that simple type of observation goes a long way, as we shall see. He also had no precise, accurate, and universal means of telling the time of or measuring the durations of events (such as eclipses). Water clocks and hour glasses and sun dials are plainly of very limited value in this regard. Doing astronomy with Ptolemy can therefore feel like fighting with our hands tied! But while we will try to understand how he got so far with so little, we will also sometimes "cheat" in order to see for ourselves how right he was about many of the basic facts on which he will base his theories. We should not deny ourselves the use of binoculars and watches when these prove useful, for instance.

Besides the *Almagest*, the book which is the focus of this course on Ptolemy, he wrote other books on astronomy and also on geography. His book *Guide to Geography* was largely a table of latitudes and longitudes of various places in the "inhabited world" (or in the world known to be inhabited in his time), and because of it he was almost as famous for his geography as for his astronomy.

The original title of his monumental astronomy book was rather uninspiring. It was called "The Mathematical Composition." But it came to be called "The Great Astronomer," and then the Arabs called it "The Greatest," combining the Arabic prefix *al* to the Greek *megiste*, and ever since it has been called "The Almagest."

Like Euclid's *Elements*, it consists of 13 books. Chapter One of Book 1 is a preface in which Ptolemy places astronomy among the other sciences in accord with his understanding of their classification, and a word or two about the great dignity of the study of the stars.

THE ORDER OF *THE ALMAGEST*. In Chapter Two of the Book 1 Ptolemy explains the overall order in the parts of his book, which is roughly as follows:

(1) First, in Books 1 and 2, he determines the ratio and situation of the earth to the whole of the heavens (or universe). This is like a general understanding of how we fit into the whole universe.

(2) After that, in Books 2 and 3, he takes up the position of the "ecliptic"—don't worry about that just yet, we'll get to it. He will also explain how to determine where you are on the earth and how certain celestial appearances vary according to your location on the earth (i.e. your latitude and longitude). You can't really get very far without this information. It amounts to understanding your own point of view, your location as an observer and the properties of your place of observation.

(3) In Books 3 through 6 he develops the theory of the solar and lunar movements and their eclipses. The prediction of lunar and solar eclipses was of central importance to ancient astronomers.

(4) Books 7 through 13 were devoted to the stars. Books 7 and 8 were about the so-called "fixed stars", which we would simply call "the stars" today. He catalogued constellations and the like. Books 9 through 13 were about "the wandering stars", that is, the planets (the word "planet" comes from the Greek word for "wanderer"). These five "wandering stars", namely Venus, Mercury, Mars, Jupiter, and Saturn, don't keep their positions among the other stars, the "fixed" ones, which all stay still relative to each other (so far as naked eye astronomy is concerned!).

Note the order in which Ptolemy proceeds in his astronomy generally respects the following principles:

• He goes from what is at rest to what is in motion, and from what has fewer motions to what has more motions. So he considers the Earth first, and the heavens only generally by contrast, before coming to the particular motions of things in the heavens. And he considers the "fixed" stars first, since the "wandering" stars have all the motions that the "fixed" stars have and more motions in addition.

• He goes from what is close to us to what is further away (e.g. he takes up the sun and moon before the fixed stars, our own location on earth before the sun and moon). He does talk about the sun before the moon, but that is because the moon is less regular in its motion. Similarly, although he talks about the fixed stars before the planets, that is because there is less for him to say about the "fixed" stars, since their motions appeared much more regular than those of the "wandering" stars.

• He also goes from what is easier to observe to what is harder to observe. Hence he takes up the Earth first, the Sun next, the Moon next, then the fixed stars, then the wandering ones. The wandering stars or planets are in some ways easier to observe than the fixed ones, insofar as they are generally brighter. But it takes many more observations to acquire any power of predicting where they will be in your sky at 6pm on some particular evening two years from now than would be required to make similar predictions about a "fixed" star. We will see this soon enough.

Ptolemy next subdivides the first part of his *Almagest*, concerning the ratio and situation and condition of the earth in relation to the whole of "the heavens", i.e. in relation to that whole world of stars out there. Here is what he intends to show, in order:

(i) That the heavens are spherical and move spherically.

(ii) That the earth, in shape, is "sensibly" spherical.
(iii) That the earth, in position, lies at the center of the universe.
(iv) That the earth, in size, has the ratio of a point to the universe.
(v) That the earth has no local motion at all.

In short, he is arguing for a round earth and for a geocentric view of the universe. Notice he is *not* arguing for a *flat* earth! It is a myth that everyone believed the Earth was flat until Christopher Columbus proved the opposite. Christopher Columbus proved no such thing, and many before him did prove it, including Ptolemy.

We will return to these first five Ptolemaic Propositions in Day 2. But for the remainder of today, we will consider some basic back yard astronomy exercises in order to orient ourselves under the sky, and become familiar with what it means to be an observer of the heavens. Ptolemy assumes we have this basic understanding, but most people don't—not today any more than in his day. Not all of these exercises can be accomplished in a single night, or even in a single week or month. If possible, you should continue those which require more time throughout your reading of this course. What you should find is explained in the notes following the exercises.

WARNING: The exercises are extremely INCONVENIENT! For that reason, I will not assume that you are willing to go through with them. If you have the will, then by all means, make the observations for yourself. But if you do not, read through what they ask of you, and see if you can guess what it is you should observe—and then read the subsequent explanations (included below the exercises as lists of "Phenomena") of what you in fact would observe, if only you took the trouble. But do not skip the exercises altogether, at least not if you are a beginner. Little of what follows after will make much sense to you if you do.

EXERCISE 1

Go outside on a clear night, as far away from "light pollution" as you need to get in order to see at least some stars clearly. Pick certain conspicuous stars, and pick one spot on the ground or in a chair from which to view those stars over the course of several hours. You need not watch them continuously, but pick something like a tree or building or mountain to compare their location to, so that you can see whether they are moving and in what direction. Do all the stars move? Do they somehow move together? What seems to be the shape of the path of each star?

Several times during the course of one hour, note the positions of several bright stars which are nearly overhead. In what direction do the stars appear to move?

EXERCISE 2

If you know you are in the northern hemisphere, find Ursa Minor, also known as "The Little Dipper." Watch what that dipper does over the course of several hours, and describe it. If you happen to be in the southern hemisphere, find Octans or Hydrus or Apus, and describe what these constellations do in the course of several hours.

EXERCISE 3

Observe some star which sets at a time convenient for you to observe. It's all right if you cannot observe it setting on the true horizon. If you can watch it disappear behind a distant mountain or hill, that will do—but not behind some nearby or mobile object, like your own hand! Always observing it from exactly the same spot, and with your eye in precisely the same spot (you must use some sort of fixed object to determine this, like the edge of a railing you can use as a line of sight), record the exact time of its setting at least 5 nights in a row. Is its setting time always the same? Does it set at the same place on the horizon every night?

EXERCISE 4

Observe where the sun sets on your horizon at least every third day for two to three weeks. You must again take care to observe this with your eye looking along exactly the same line of sight and from the same location each time. You should make a record of the locations of the settings, perhaps by drawing fixed distant landmarks, such as mountains and trees, and marking where the sun set in reference to these. If at all practical, see if you can do this for a full year, with dates above each location of the sun. What pattern can you discern in the settings?

EXERCISE 5

Find a place from which you can observe as much of the true horizon as possible, and from which you can observe the sun rising above and setting below this horizon (rather than some higher "horizon" like the ridge of a mountain range). Go to this place and observe the sunrise and (on the same day) the sunset and record the times. This will give you the length of the day on that day of the year in your location on earth. Do this at least once a month. Are the days unequal? If so, around what time of year does the day appear to be longest? When is it shortest? Are there days of equal length?

EXERCISE 6

Find some constellation that rises in the east soon after the sun sets in the west. Every third evening or so, for about 3 weeks, record the exact time of the setting of the sun (again taking precautions to do this from the same spot, and with the same line of sight each time), and, facing about, record the exact time of the first star to rise in the constellation you are observing. Do the times stay the same? Does the interval between the sun-set and the star-rise remain the same? What kind of pattern or movement do you seem to be observing?

EXERCISE 7

Observe the moon for five successive nights, beginning as soon as possible after the new moon (the dates of the moon's phases can be found on many calendars). Each night, note in which constellation it lies, and near which particular stars. Is the moon always in the same place relative to the stars? If not, how does it move? Each night, note the time of the moon's setting. Does it set at the same time each night? If not, how does it change?

PHENOMENA RELEVANT TO EXERCISE 1

Stars directly overhead appear to move from east to west. If you are facing due north (or due south in the southern hemisphere), and you wave with your right arm over your head, the stars move that way.

The path of each star is a circle, but that is not easy to discern with any degree of exactness unless you can find the "poles" of the stars' motion. They all move together in concentric circles about the north and south "celestial poles," the two spots in the night sky that *don't* move. If you live in the northern hemisphere, you will see only the north celestial pole in your sky; if in the southern, you will see only the south celestial pole in your sky. If you live at the equator, you will see both at once, each one lying right on your true horizon. We will come back to these things more distinctly as we get back into Ptolemy.

PHENOMENA RELEVANT TO EXERCISE 2

The star at the end of the handle of "The Little Dipper" is Polaris, the North Pole Star. The Little Dipper itself moves counter-clockwise around Polaris, like a clock-hand going the wrong way. All stars in the northern hemisphere move counter-clockwise around Polaris. Polaris itself basically sits still—not exactly, since it does not sit exactly at the North Celestial Pole, but it is fairly close. So really, it makes a tiny circle up there around the true North Pole. If you took a very circular tube (like a telescope-tube!) and fixed it on a stable tripod, and centered it on Polaris, then, looking inside the tube, any star which was on the edge of your circular field of vision would go exactly around that circle counter-clockwise. So the motion is quite circular.

If you live in the southern hemisphere, then the stars go in concentric circles *clockwise* around the South Celestial Pole. There is no easily visible star very near the South Celestial Pole—so there is no southern analog to our Polaris in the northern hemisphere. But with care, you can still find that spot in the night sky that doesn't move—it is not far from Sigma Octantis, if you can find that naked-eye star.

Does it make sense for there to be two poles for the motion of the stars? Of course. The heavens can be thought of as a big ball up there (remember, we are soon going to start off as geocentrists with Ptolemy!). When you spin a ball, there are two spots on its surface that don't move, namely the ends of the axis of its rotation, the one diameter that stays in the same location. If you were inside the ball (on a little planet in the middle), and the rotation about one pole looked clockwise to you, then the same rotation about the other pole would look counterclockwise to you. That is a sample of the kind of imagining one must do in order to grasp the art of astronomy!

PHENOMENA RELEVANT TO EXERCISE 3

You will observe that a given star sets about 3 minutes and 55.91 seconds *earlier* each night (but this is regular year round). So one "sidereal day" (i.e. the time it takes a star to go around the earth) is shorter than a 24 hour day by that much, i.e. one sidereal day = 23h 56m 4.09 seconds.

The setting place of a star is basically the same every night (or during the day, if that's when it is in the sky for us). This is because the fixed stars do not change their location relative to each other (ignoring the way they drift about in our galaxy, and thus change the constellations, but too slowly for us to see by the naked eye during a human lifetime), and again because the celestial poles do not move in our sky (ignoring the precession of the equinoxes, or the swivel in earth's axis, which has a period of about 25,000 years, and which we will talk about later!).

PHENOMENA RELEVANT TO EXERCISE 4

The sun, however, does *not* set at the same place on the horizon each night, but will set more and more to the north between late December and late June, and then, turning around and going the other way, it will set more and more to the south between late June and late December. And there will be a definite range of all its settings on your horizon. You will never see it set due north, for example. So there is a range of places it can set (and, conversely, rise) on your horizon, but it is a definitely limited range.

PHENOMENA RELEVANT TO EXERCISE 5

If you happen to live at the equator, all your days and nights will be 12 hours. If you live anywhere else, you will have unequal days and nights. Let's suppose you don't live very close to the arctic or the antarctic circles (we will talk about those later). Then, if you are in the northern hemisphere, your longest day (and shortest night) will occur around June 20 or so. Your shortest day (and longest night) will occur around December 20 or so. And your longest day will be equal to your longest night! Two days will be equal in length: around March 20 and around September 20. On those days, you get a 12-hour day and a 12-hour night. But the length of your longest day (or, what is the same, the length of your longest night) depends on your latitude, i.e. on how far away you live from the equator. We will see more about that with Ptolemy's help later.

PHENOMENA RELEVANT TO EXERCISE 6

We have seen that each night the stars set (or rise) almost 4 minutes earlier than 24 hours from their last setting (or rising). But each night the sun sets an average of 24 hours from its last setting. That means the time separating the sun's setting, and the setting of some particular star, gets *less* every night. If the sun sets at the same time that some star S rises, what will happen in 24 hours? The sun will be setting, *but that star will already have been above the horizon for almost 4 minutes*. Effectively, the sun not only has a daily motion in the same direction as the stars, but it also has a "lagging behind" motion, so that it is gradually creeping eastward among the fixed stars! Plato called the daily, obvious motion of the Sun "the motion of the same," the general motion it has in common with all the other stars, and he called its backward, eastward motion "the motion of the other," its distinctive motion. We can also call these the "daily" and "yearly" motions, since it takes only one day for the Sun to go about the Earth (remember, we must get ready to think Ptolemaically!), but it takes a whole year for the Sun to creep all the way back through the stars once, and get back to its same spot among the stars that it was in before.

If we plot the Sun's "backward" yearly motion eastward through the constellations, it turns out that it makes another circle up there, a circle which is *not* parallel to all the concentric circles about the two celestial poles. Although we will introduce it again later, we might as well note here that this circle of the Sun's yearly backward path is called the ECLIPTIC. (Probably it has this name because it is only on this circle that solar and lunar eclipses can occur, because the Sun is always somewhere on that circle, and we, as we shall see, are at the center of it.) Moreover, it will turn out that the moon and the 5 planets travel roughly along that same circle, and also have their own “backward” motions among the fixed stars.

PHENOMENA RELEVANT TO EXERCISE 7

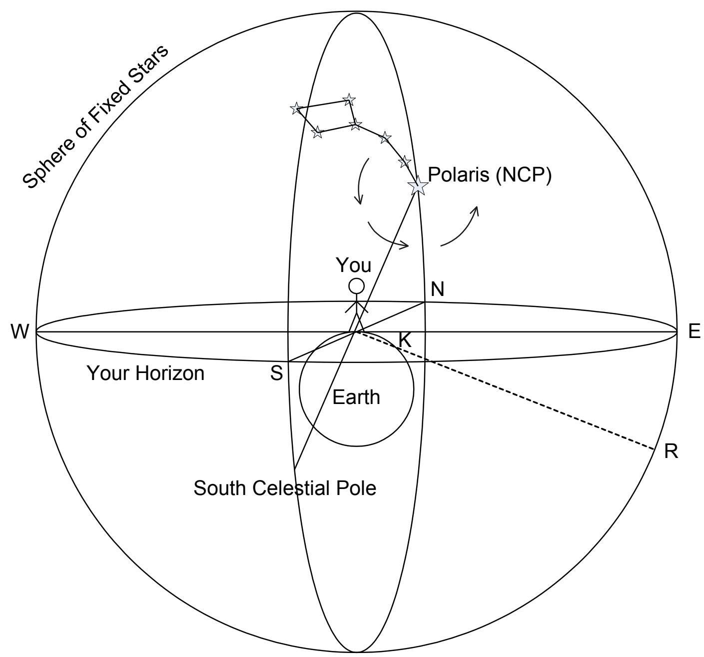
The moon sets about 40 to 50 minutes *later* each night. But the stars set about 4 minutes *earlier* each night than their last time of setting. So if the moon set together with star S tonight, tomorrow night star S will be setting and the moon will be 40 or 50 minutes behind S, which is to say the moon will be that much further *east* of star S the next night. Consequently, the moon, like the Sun, is also creeping backward through the stars (and also roughly along the ecliptic), which is to say eastward, although much faster than the Sun—the moon completes one such circuit in about 1 month (27.3 days).

PTOLEMY

DAY 2

THE BASICS OF THE GEOCENTRIC MODEL

DIAGRAM.

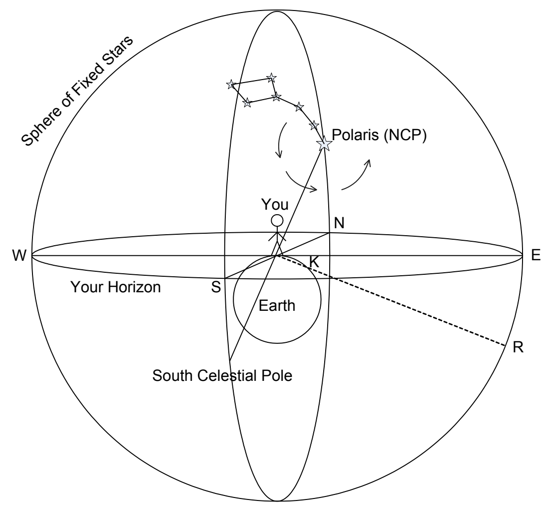


The Exercises in Day 1 contain some of the most basic observations at the foundation of Ptolemaic astronomy. There was no diagram to accompany them because it is good to begin from raw data. That way, one can distinguish pure observation from the model produced for the sake of understanding it. We will now begin to introduce diagrams and to try to account for the facts, beginning with just the daily rotation of the stars, and working our way to the Sun's "backward" motion along the "ecliptic", and eventually to the planetary motions, introducing new observations and phenomena along the way, as needed.

The first diagram depicts the Earth, and a very large you (for clarity) standing upon it. Suppose you are in the northern hemisphere, and you are looking at Polaris in the night sky, the tail end of the "Little Dipper" (or Ursa Minor). We draw an imaginary plane under your feet where you stand, tangent to the surface of the Earth (which we depict as a sphere, although we will argue for that shortly), and that plane is extended till it meets the "sphere of fixed stars", at whose center the Earth sits, immobile. This celestial sphere has all the "fixed stars" fixed upon it, and as it rotates, we see them all make counter-clockwise concentric circles about P, Polaris (or, more precisely, about the North Celestial Pole, which is very near Polaris). We have not yet argued for any of these ideas, but these are the basic elements of the Geocentric Model.

The circular plane under your feet, tangent to the Earth, is called your "**horizon**." It extends out as far as the stars. If the stars lie on a sphere, then the line which forms your horizon is a circle on that sphere. More than that, if the Earth is of insignificant size compared to the sphere of fixed stars (its size is exaggerated in the diagram just for the sake of clarity), and if the Earth sits at the center of the sphere (as Ptolemy will propose and defend), then your horizon is a "**great circle**" on the celestial sphere. A "great circle of a sphere" is a circle whose center is the same as the center of the sphere on which it lies, and so it is a circle of the largest possible size on that spherical surface. Not every circle on a spherical surface is a great circle. For instance, the circles which the stars in the Little Dipper make around Polaris are not great circles, although they are circles on the sphere of fixed stars. Their centers lie closer to Polaris than to the center of the sphere itself.

Now let's familiarize you with some of the properties of your horizon. First of all, you can see everything above that plane, but nothing beneath it. Choose any star R on the sphere of fixed stars which lies below that plane, and join R to you by a straight line. Then that line has to cut the sphere of the earth at some point K. That is a basic fact of spherical geometry. The line joining you to K must lie entirely inside the Earth, another fact of spherical geometry. Hence there is no way for you to see R, since a big chunk of solid earth lies between it and your eye.



The diagram might make it seem as though you are lucky to be seeing fully half of the sphere of fixed stars, since you happen to be standing at the center of the celestial sphere, whereas other people on Earth would not be, but would be standing some terrestrial distance away from you, and hence away from the center of the sphere of fixed stars. But since the Earth (as we shall see) has no significant size compared to the celestial sphere, those terrestrial distances make no real difference. It is as if everyone were standing at the center of the celestial sphere. Imagine growing the Earth inside the diagram (but put its center at the center of the celestial sphere) so that it becomes almost as big as the celestial sphere itself: Then you will get scrunched up near the top of the celestial sphere, and your horizon will cut off a very tiny portion of the celestial sphere, and so you will see very little of it; much, much less than half. Conversely, if you imagine shrinking the Earth till it is a microscopic speck in the diagram, sitting right at the center of the celestial sphere, your horizon will cut off pretty much exactly half the celestial sphere. And whether we think there is a celestial sphere or not, we can show that the size of the Earth is insignificant compared to the distances out to the stars (as we shall see).

Now consider the point on your horizon directly below Polaris, and call that N. The line joining you to N points to terrestrial North. If you were to walk in that direction long enough, Polaris would keep moving up in your sky until eventually it was directly overhead and you were standing on Earth's North Pole. (Don't try that.) Extend the same straight line from N behind you, now, and that will point due South on your horizon. Directly to your right is East, and to your left is West, if you are facing Polaris. As we noted in Day 1, all the stars, as well as the Sun and the Moon, rise somewhere on the Eastern portion of your horizon, and move together in apparently concentric circles around Polaris, and set somewhere in the West. If we extend the line "from Polaris to you" below your horizon, through the Earth, and out to the celestial sphere, that point is the South Celestial Pole. This straight line joining the North Celestial Pole (near Polaris) to the South Celestial Pole is the "axis of the daily motion," the axis around which the whole celestial sphere spins once every 24 hours (roughly). Strictly speaking, that axis goes not through you, but through the center of the Earth. But since you are an insignificant distance away from the center of the Earth, compared to the distances out to the stars, it makes no difference which we say, so far as the stars are concerned.

That is our first diagram, and our first taste of the Ptolemaic geocentric model of the universe. Soon we can begin to bring forward some of Ptolemy's arguments in favor of it.

SCHOLIUM ON THE FOUR POINTS OF THE COMPASS

How are North, South, East, and West defined?

"Celestial North" means the spot in the heavens that does not move, near the pole star. On earth, the place called "Terrestrial North" means the spot where Celestial North is directly overhead. (Do not confuse this spot with magnetic north, which is not exactly the same!) On earth, if you can see Celestial North (Polaris), but it is not directly overhead, the direction called "North" means the direction toward the place on your horizon directly below Celestial North in the sky. If you cannot see Celestial North, then you can see Celestial South, in which case the spot directly beneath it on your horizon is the direction called terrestrial "South," and the opposite direction is "North."

"West" = directly to your left as you face north.

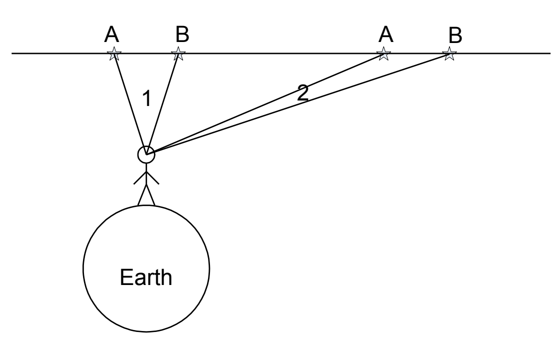
"East" = directly to your right as you face north.

PROPOSITION 1: THE HEAVENS MOVE SPHERICALLY.

In Book 1, Chapter 3 of his *Almagest*, Ptolemy argues that the heavens move spherically and uniformly. In these arguments, one is presuming that it is the heavens, not the Earth, that is moving. The arguments do not prove this, but assume this, and go on to prove that the motion of the heavens must be spherical and of uniform speed (so far as the unaided senses can determine).

[A] If you pointed a perfectly circular tube directly at Celestial North, through the tube you would see the stars on the edge of the tube trace out the circular shape of the tube. They would not stray from it. And this is true no matter how big a piece of sky you can see through the tube. Also, if you use a watch (Ptolemy would have had to make do with an hour glass or a water clock), and time how long it takes the star to go through any eighth-arc of its full circle, it will take exactly that long for it to go through any other eighth-arc of the circle. So the motion is uniform.

[B] Other ideas about how they move make no sense. For example, if we supposed that the stars move along straight lines, away to infinity, then the same stars would never return. But the same stars do return. So that cannot be how they are moving.



[C] In fact, if the fixed stars moved around us by being carried along any surface other than a spherical one, they could not possibly keep the same angular distances from each other from our point of view unless they changed their actual linear distances from each other, which would be very complicated and bizarre—as though the universe were playing a trick on us, making it look as though the stars kept the same linear distances from each other, but really they didn't. To see this, imagine yourself observing two stars, A and B, at two different times. Suppose A and B are carried along on some surface other than a spherical one—say a flat surface, moving by like a conveyor-belt. At one time, the angle between A and B and your eye at the vertex (which you can measure with a graduated protractor whose arms are sighted against the stars with your eye at its vertex) is angle 1. Later, if A and B stay the same distance apart from each other, the angle between them will be angle 2, a much smaller angle as A and B get further away from you. The only way to keep the apparent angle between A and B the same, then, would be to have the distance between them increase as they got further from you. And that makes it seem as though the universe is playing a trick, just especially on you. Since that is ridiculous, we have to say instead that A and B keep the same distance from each other, as well as the same apparent angle from each other, which happens only if they are moving on a sphere.

And, Ptolemy points out, if you think the fact that the stars appear larger near the horizon (like the moon) proves they are closer to us there, you are wrong. He attributed this fact (which is at least to some extent an illusion) to the atmosphere. We are looking out through more atmosphere, and more moisture, when we see a star near the horizon than when we see it nearly overhead, and that affects its appearance.

Ptolemy gives other arguments, but these are the clearest and best of them. Note that no argument here establishes that the fixed stars are all the same distance away from us. They could be stuck in a giant crystalline sphere at different distances from us, so far as these arguments are concerned. But they maintain their distances from each other (hence they are called "fixed" stars), and always appear to have the same angle between them when we observe them with a sighting instrument.

PROPOSITION 2: THE EARTH IS SENSIBLY SPHERICAL

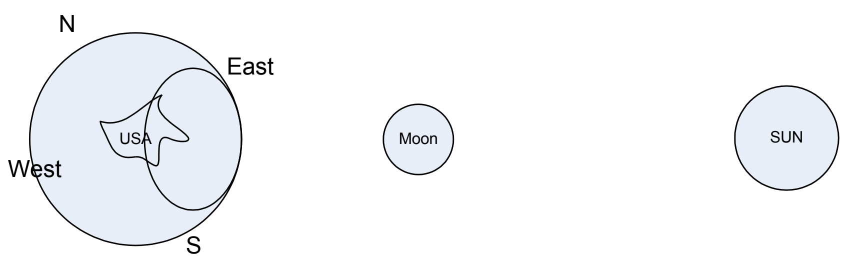
Ptolemy proves this in Book 1, Chapter 4 of his *Almagest*. Note again that he knew this long before Christopher Columbus—and it was known long before Ptolemy, too. But why does he say "sensibly" spherical? I think for two reasons. One, because you can actually see the curvature of the earth, e.g. when a ship approaches from over the horizon, first you see the top of the mast, and later the ship itself, so that the bulge of the Earth between is visible. Two, because the earth is not a perfect mathematical sphere, as we can see from its mountains and valleys, but when one gets back far enough, the whole impression is very nearly a sphere. It is truer to say that the Earth is an "oblate spheroid," a bit fatter about the equator and squashed in at the poles, but not much.

ARGUMENTS FOR THIS BASED ON LONGITUDE:

Positive Arguments

(a) Even without watches, we can know the sun rises and sets sooner in the eastern parts of the earth than in the western parts (e.g. in New England vs. California). And an eclipse of the sun is a more or less simultaneous event that can be seen from regions fairly far apart on earth (from very different longitudes). But the sun is higher in the sky for eastern folk (when an eclipse occurs) than it is for western folk. So the earth is roundish in an east-west direction. And since the further east you go, the higher the sun is in the sky proportionally for a given moment (like an eclipse), the earth is *circular* in an east-west direction.

When an eclipse occurs, for instance, the sun is right overhead for Europeans, but still rising for midwestern Americans.

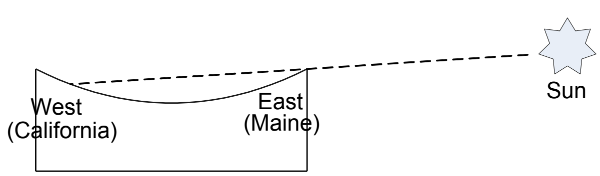


(b) Mountains on land appear to rise out of the sea as we sail in from sea toward the shore, and so the ocean is convex. This argument is independent of direction, unlike (a) which argues that earth is round in the east-west direction, or longitudinally.

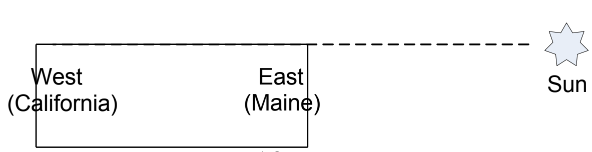
(c) An additional argument: The earth's shadow during a lunar eclipse is circular.

Negative Arguments

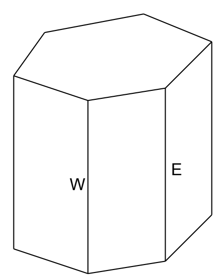
(1) A concave earth (from E-W) would mean that the people in the West would see the sun rise before people in the East. That doesn't happen.



(2) A flat earth (from E-W) would mean that stars would rise and set at the same time for people in the East and in the West. That doesn't happen.

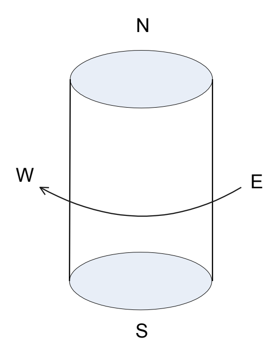


(3) A polyhedral solid earth (from E-W) would mean that stars would rise and set at the same time for any people at the eastern and western edges of a flat face on that solid. But again, that does not happen. Even at relatively small distances, e.g. if A lives 200 miles west of B, there will be observable differences in the time of the rising of the sun. Today, we can easily verify this. If you live in New Hampshire, call your friend in California just as the sun is rising for you. He will be angry with you, because it will still be completely dark where he is—the sun is above your horizon, but still below his for another three hours. Ptolemy did not have telephones, nor could he bring a watch with him and travel 200 miles or so west and just record the time the sun rose on his unadjusted-to-local-time watch. He would rely on things like eclipses, which could be observed simultaneously by people 200 miles apart, and who could observe how high up in the sky the sun was for them at the time.



ARGUMENT BASED ON LATITUDE

(4) A cylindrical earth (standing N-S) would mean there are no always-visible stars for inhabitants of the curved surface. All stars that could be seen from the curved surface would move from East to West and eventually lie behind the earth, and hence be invisible until they rose again. But, with the exception of earth's equator (where all stars move from east to west with the exception of Polaris, which sits still right on the northern horizon, or nearly so), from anywhere on earth there will be stars that are always visible, i.e. that never set (although they will be hard to see during the day, thanks to the Sun).



Also, a cylindrical earth (standing N-S) would mean there would have to be some stars which are visible from all latitudes along that cylindrical surface.

For example, Polaris would be visible from all latitudes along that surface. But in fact, it is possible to go south far enough that Polaris disappears below one's horizon (just as we cannot see the south celestial pole if we live in the northern hemisphere).

According to a cylindrical model of the earth, there would be no change in which stars were visible as we went north or south. As it is, when we travel north, certain stars in the south dip below the horizon and are never to be seen from that latitude, and new stars become visible above the northern horizon that were not visible where we were before. As Ptolemy puts it: "the more we advance towards the north pole, the more the southern stars are hidden and the northern stars appear."

PTOLEMY

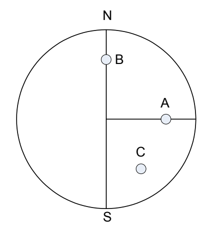
DAY 3

MORE BASICS OF THE GEOCENTRIC MODEL

PROPOSITION 3: THE EARTH IS AT THE CENTER OF THE UNIVERSE

Ptolemy argues for this in Book 1, Chapter 5 of his *Almagest*.

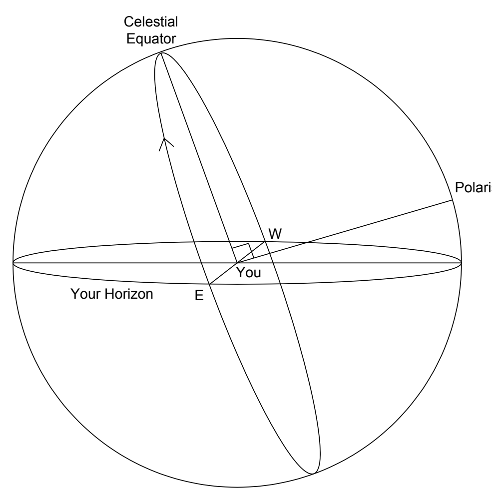
He argues like this:



If the earth is not at the center of the celestial sphere, then either:
[A] It is off the axis, but equidistant from the poles
or [B] It is on the axis, but not equidistant from the poles
or [C] It is both off the axis and not equidistant from the poles.

PROBLEMS WITH [A]

(1) Ptolemy probably never went to earth's equator. Nonetheless, he makes correct assumptions about what one would see there, and today we can argue from this more confidently, since people live at the equator, and we can call them up or travel there ourselves.

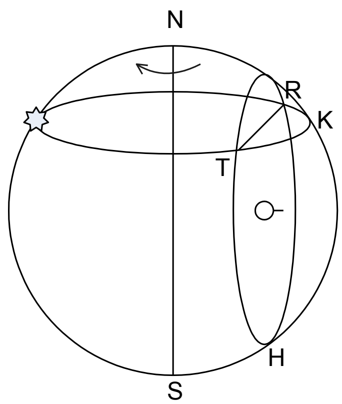


Just as the terrestrial north and south poles are defined by the celestial ones, so too the terrestrial equator is defined by the "**Celestial Equator**." The celestial equator means the great circle in the sphere of fixed stars which is at right angles to the axis of daily rotation, i.e. to the diameter passing through the celestial poles. To see where the celestial equator is in your sky, make an "L" shape out of two straight rods at right angles to each other. (You can just imagine this, you don't actually have to do it.) Point one leg of this rod at Polaris, i.e. at Celestial North. The other will point somewhere up in the sky. Twist the rod pointed at Polaris, and the end of the other rod will be tracing out where the celestial equator is in your sky.

If Polaris is above your horizon, going 90° straight overhead from it will clearly take you beyond your "**zenith**", that is, beyond the point in your sky straight overhead. So the celestial equator will not be at right angles to your horizon. But if Polaris is right on your horizon, then going 90° from it in a direction perpendicular to the horizon will take you right to your zenith, and so the celestial equator will be at right angles to your horizon. And in that case, you live somewhere directly under the celestial equator, which is to say, you live on Earth's equator, the terrestrial equator.

Ptolemy calls this view of the heavens, the one that you have at the equator, the "right sphere." That means the spherical motion of the heavens is at a right angle to your horizon, instead of coming up from your horizon at a slant. Most of us, however, live in an "oblique sphere," a place where the motion of the heavens comes up from our horizon at a slant, and so the celestial equator is not straight overhead, but it is to some degree oblique to our horizon.

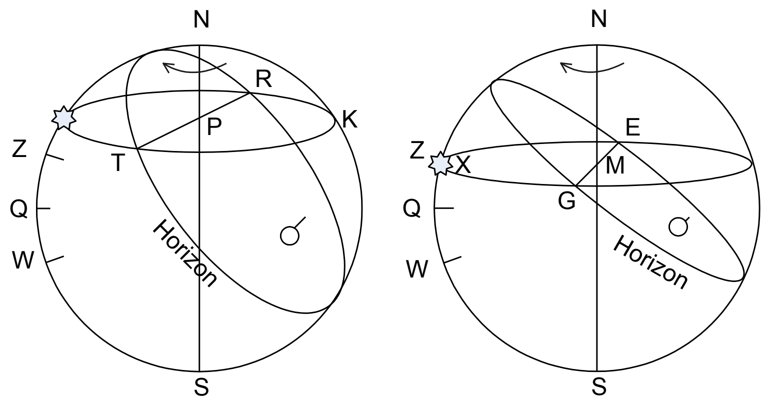
With those terms in place, we can now argue against option [A].



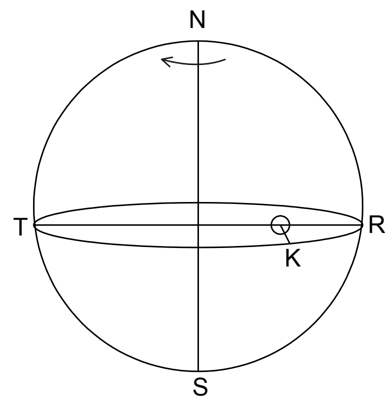
If we try to suppose that the earth lies off the celestial axis somewhere, as depicted in the accompanying figure, what would that do to the appearances for someone living on earth's equator? Suppose you live there, and we draw you standing up on the earth as the little tiny stick to the right of the little earth. You will *never* get an equinox if earth is off the axis, since the sun rises at R, culminates at K, sets at T, and so daylight is arc RKT, which is much less than the night arc below your horizon. The truth is that *every* day is an equinox in the right sphere, i.e. at the equator.

So option [A] is obviously impossible.

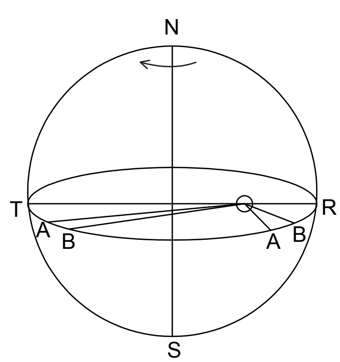
(2) In the "oblique sphere" (i.e. anywhere else), either you *never* get an equinox, because the one place where the sun's path would be cut in half by the horizon, namely RKT, is beyond the limits of the sun's yearly motion (W & Z), or you get an equinox, but at a place EXG, which is not half way between the limits of the sun's yearly N-S motion (contrary to the facts).



(3) Again, if the earth stood away from the celestial axis, then, anywhere on earth, e.g. at earth's equator, the time from star rise (R) to star culmination (K) would be much less than the time from star culmination (K) to star set (T). And stars would appear much bigger at R than at T. And none of that is observed.



And if I am in the right sphere, and TNRS is my horizon (I am standing on the little earth in the diagram, and you are looking straight down at my head), then the angle between stars A & B will differ as they go west. But that is not observed. (Note that these arguments assume what we argued for earlier, namely that the stars are moving by being carried along with a giant celestial sphere.)



PROBLEMS WITH [B]

Even if we put the earth on the axis, we run contrary to many observations if we shift if off-center. To see this, let's start by introducing another term, the "**zodiac**." We saw on Day 1 that the Sun creeps "backward" (eastward) through the fixed stars, taking a year to complete one such circle. That great circle on the celestial sphere is called the "ecliptic," and it has a fixed location among the stars. More exactly, the ecliptic is not the circle which the Sun is actually on, but the one it appears to move on from our viewpoint—it is the circle of the Sun's apparent motion in the fixed stars. (The Sun is actually a lot closer to us than the fixed stars, and Ptolemy knew this.) The ecliptic, in other words, is a *projection* of what the Sun seems to do around us onto the distant backdrop of the fixed stars. So we can plot out, on any chart of the stars, the line along which the Sun makes this slow, backward crawl, and you will in fact see such a line labeled "ecliptic" in most star charts. You might wonder how we can do this, since the Sun is so bright it makes it impossible to see the stars anywhere near it. How can we say where it is among the stars? Well, we can plot the stars at night, of course, and throughout the year, as the Sun creeps eastward, we gradually see different stars at night. So it is that we have "summer" constellations and "winter" ones. After a year of careful charting, we will have seen all the stars that the Sun moves through (and this work is already done for you on any star chart). To see which stars the sun is among right now, we need only look at which stars are rising in the east just as the sun is setting in the west, and then look at our star chart and see which stars are 180° away, or directly opposite, those rising stars. Those opposite stars will be the ones that the Sun is in right now.

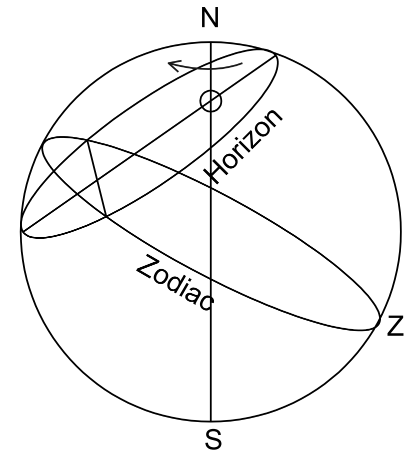
It so happens that the other "wandering stars," the five naked-eye planets (Mercurcy, Venus, Mars, Jupiter, Saturn), and even the Moon, also "creep backward" roughly in the vicinity of the ecliptic, although they do not quite stick to it as the Sun does. All these celestial bodies are always to be found within about 8° (with our eye at the vertex of the angle, as always) north or south of the ecliptic. So there is a whole belt up in the night sky, 8° on either side of the ecliptic, which is very important to watch because all the planetary and solar and lunar stuff happens in there. This belt is called the "zodiac." It is divided into 12 equal portions, each one being 30° (thus totalling the 360° of the zodiac). Each of these portions is called a "**sign**," and is named after the principal constellation within it. They began (by convention) at the spot where the Sun was on the day of the spring equinox (more on equinoxes later). The twelve familiar "signs of the zodiac" are:

Aries	(the Ram)
Taurus	(the Bull)
Gemini	(the Twins)
Cancer	(the Crab)
Leo	(the Lion)
Virgo	(the Virgin)
Libra	(the Balance)
Scorpio	(the Scorpion)
Sagittarius	(the Archer)
Carpricornus	(the Goat)
Aquarius	(the Water Bearer)
Pisces	(the Fishes)

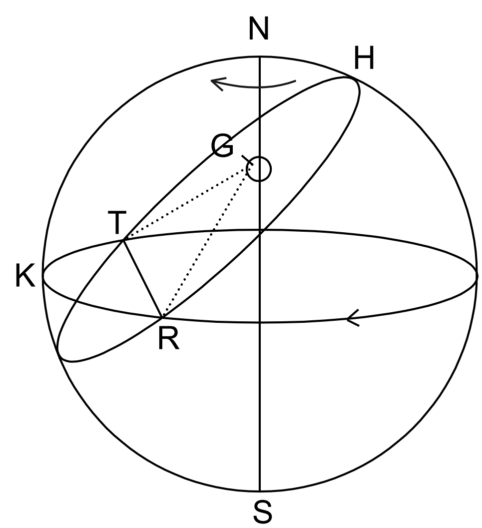
Apparently it is because most of these are animals that the “zodiac” got its name from “animal-circle” in Greek, or some such combination.

Now let’s get back to the argument against [B], that is, against the idea that the Earth might be on the celestial axis, but off-center.

(1) If you are anywhere besides the right sphere (since in the right sphere the horizon would still divide the zodiac into equal parts), your horizon would cut the zodiac into unequal parts. But we can always see 6 signs, no matter where we live, and 6 signs = 180° on the sphere (which we know because any 6 signs can fill our sky, so any 6 signs are equal to any others in angular length, so any 6 must be 180°). So our horizon in fact cuts the zodiac into equal parts. So it is not possible for the Earth to be on the celestial axis, but off center.



(2) Again, suppose it is the day of an “**equinox**”, a day of equal daylight and darkness (12 hours each). That happens when the Sun is on the celestial equator (the ecliptic, the great circle that the Sun is *always* appearing to move on, cuts the celestial equator at two points). If Earth is on the celestial axis, but off-center, as in the accompanying figure, an equinox will occur when the Sun’s backward crawl brings it onto the celestial equator, so that on that one day it rises at R and sets at T. If we drive a stick into the ground, the shadow cast by the Sun at R at sunrise, and the shadow cast by the Sun at T at sunset, will form an angle, and not lie in a straight line. But that is not what we observe. The opposite in fact occurs; in the oblique sphere, on the day of an equinox, the shadow sweeps out 180° during the day (so it starts and ends in the same straight line, though pointing in opposite directions). In the right sphere, on the day of an equinox, the shadow stays in one straight line all day (the sun goes directly overhead, just as the celestial equator does there).



PROBLEMS WITH [C]

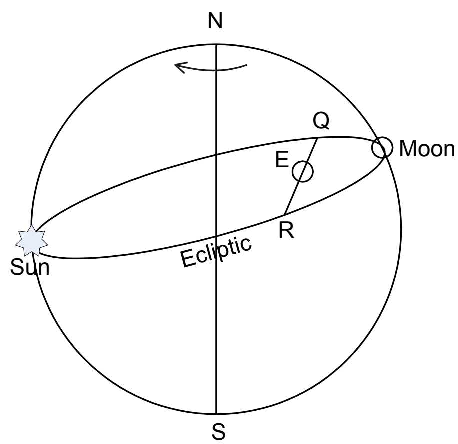
Alternative [C] was to put the Earth *both* off the axis *and* closer to one celestial pole than the other.

This idea has all the problems of [A] and [B] together, and so it must also be rejected.

An added problem concerns lunar eclipses. Imagine the celestial sphere as a giant basketball, spinning about the axis NS. The equator is a great circle at right angles to NS, and so if we drew that on the basketball, it would appear to sit still, rotating in place. But the ecliptic, as we saw, although it is a great circle, must be tilted on the surface of the spinning sphere, since by creeping on it the Sun is sometimes north of the equator, sometimes south of it—which is confirmed by our initial exercises in observing the location of sunsets on our horizon throughout the year. The ecliptic then, would appear to be "wobbling" on our basketball, not just spinning in place or within itself like the equator.

This means that if the earth rests somewhere off center, then there would be only one position of the ecliptic which would ever pass through the earth, and so only there could a lunar eclipse occur. That would mean there could be a lunar eclipse only in two signs, namely in the one the moon is in (in the accompanying figure), and also in the one the sun is in (if the moon and sun switch places). But that is contrary to the facts: lunar eclipses can occur throughout the zodiac.

In the false figure drawn here, lunar eclipses could also occur at places like Q & R, but then the Sun and moon would not be 6 signs apart during such an eclipse, which again is contrary to the observed facts.



There you have it. If the Earth is thought of as sitting still, and we therefore attribute all the apparent motions of the stars to the stars themselves, then the appearances force us to admit that the fixed stars move spherically (and then the most natural reason for this would be that they are all fixed in a single, solid, common sphere), and also that we are at the center of the celestial sphere.

Notice that one reason Ptolemy did NOT give for saying we are at the center of the universe is "because we humans are the most important thing in the universe". Far from it.

He thought the heavens were somehow divine, and we were mere mortals. Plato and Aristotle thought that the heavens were living and intelligent beings, moving themselves, and that they were immortal. Aristotle thought they had always existed, had never come to be, and would always be, and they were made of indestructible materials quite unlike anything on Earth. These ideas are very foreign to us (and incorrect), but they were reasonable to some extent given the idea that the heavens were spinning all by themselves and in perfectly circular patterns. At any rate, to suppose that the Earth itself is spinning once a day is really the alternative, and that is a fairly counter-intuitive idea, as we shall see. Most of the ancients could not stomach that idea. But they did not place us humans in the middle because they thought we were so important. On the contrary, most ancient thinkers considered the center of the universe the "cosmic dump" for heavy and corruptible bodies, as opposed to the divine and celestial bodies above and in control of everything down here.

PTOLEMY

DAY 4

REMAINING BASICS OF THE GEOCENTRIC MODEL

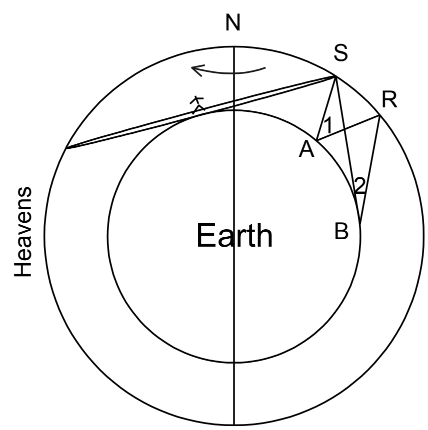
PROPOSITION 4: THE EARTH HAS THE RATIO OF A POINT TO THE HEAVENS

In Chapter 6 of Book 1 of his *Almagest*, Ptolemy proves that the earth has "sensibly" the ratio of a point to its distance from the sphere of fixed stars.

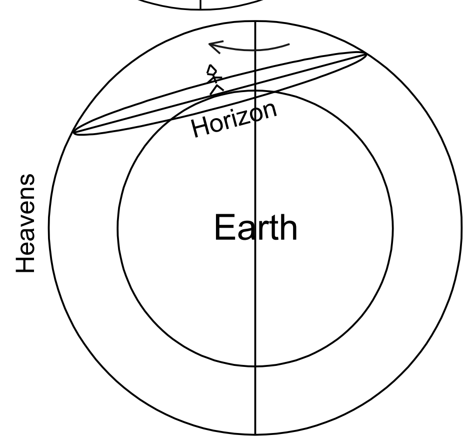
"Sensibly" means there is no appreciable difference between the actual ratio of the earth to the heavens and the ratio that a point would have to the heavens. There is no difference that has any effect on how things appear to us.

His arguments for this conclusion are as follows:

[1] If earth had significant size relative to the sphere of fixed stars, the star S would appear brighter to the observer at A than to the observer at B, since AS < BS. Also, the angular distance between stars S and R would appear different to observers at A and B, since angle 1 is less than angle 2. But such differences are not observed. So it cannot be that the Earth has any detectable size compared to the heavens.

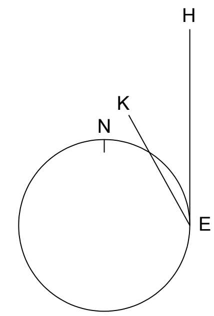


[2] If we measure time with a sundial or take observations of angles with armillary spheres on the assumption that our instruments are at the center of the heavens (although in fact we are in fact distant from it by the radius of the earth), we don't run into problems. So the radius of the earth must be an insignificant distance away from the center of the heavens.

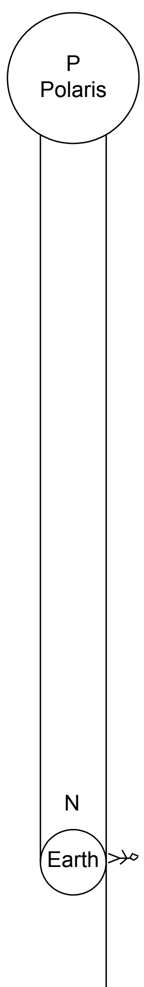


[3] If earth had significant size, our horizons would never cut the celestial sphere in half, as they in fact always do.

SCHOLIUM ON THE VISIBILITY OF POLARIS FROM THE EQUATOR



Polaris is not exactly at the North Celestial Pole, but makes a little circle around it. Still, even if it were right on that Pole, it would be visible from anywhere on Earth's equator, as a star right on the horizon (as long as one could get a view out to the true horizon). Why? If I am standing on Earth at E, on the equator, my horizon being EH, tangent to the earth at E where I stand, any line below that horizon, such as EK, would have to cut the spherical Earth, and hence block my view. To see Polaris, then, that star would have to bulge a bit above my horizon. But then it seems like it would not bulge above the horizon for someone on Earth's equator, but standing on the *opposite* side of the earth, or nearly on the opposite side. And yet he could also see it just on his horizon. How is that possible?



Since Polaris is so far away, yet visible, it must be much bigger than the earth. Hence the portion cut off by earth is so slight, you can still see practically half the star above your horizon, even if the star is exactly centered on the North Celestial Pole. Even in the accompanying figure, the size of Earth is greatly exaggerated in comparison to Polaris, and the size of Polaris is greatly exaggerated in comparison to the distance separating Polaris and Earth.

PROPOSITION 5: THE EARTH IS AT REST

In Chapter 7 of Book 1 of his *Almagest*, Ptolemy argues against the idea that the Earth could in any way be in motion, for instance, that *it* might be doing the daily spinning instead of the heavens, or that *it* might be annually creeping about through space rather than the Sun. Since this idea is false, his arguments for it will clearly not be decisive. And yet it will be instructive in each case to consider precisely where they go wrong. We will return to this question more thoroughly when we come to Copernicus. For now, let's take a look at some of the arguments which Ptolemy musters for the immobility of the Earth:

ARGUMENT 1. If the earth moved away from the center, then the same things, contrary to the appearances, would follow as in Proposition 3, which showed the impossible consequences of placing the Earth off center. So the Earth cannot move off center. So it cannot move.

Note one weakness in this argument: we have seen that the size of the Earth is insensible compared to the distance out to the "fixed" stars. But what if the size of Earth's orbit around the Sun were also insensible compared to that distance? Then we could move vast distances away from the Sun by terrestrial standards, yet these same distances would be negligible or nonexistent by galactic or universal standards! Then, for all his arguments showed before, it might be the *Sun* which is at rest in the center of the universe, while Earth makes a large orbit around it every year. On that showing, the Sun would *appear* to creep around us on the ecliptic, but then the "ecliptic" would really be a projection of *Earth's* orbit projected out to the backdrop of the stars.

ARGUMENT 2. All weights move toward the earth—an indication that it is at the center, the bottom of the world. Where else would heavy things go but "down"? And we do not see anything heavy falling toward the stars, falling "up". Hence Earth must be the "downest" place in the universe—which puts it in the middle. And how can it move from there, if it is made of heavy stuff that likes to be in the downest of all places?

Here the weakness is subtler. We have all grown up with the idea that Earth is just one of many places that has its own "down." Things fall "down" on the Moon. In fact, every planet or star has its own "down." And each planet is in a sense "falling down" toward the Sun. But these things are far from obvious just from looking outside! If the Moon is heavy, and is "falling down" to Earth all the time, why doesn't it smash into the Earth? If it were heavy, wouldn't it be heavier than a mountain? Wouldn't it come crashing down on us? But it's been up there for at least thousands of years. So it is understandable that Ptolemy and all others in his time would fail to consider that the Moon might be heavy, and that anything in the heavens might be heavy or be a place to which heavy stuff liked to go. There is a reason it took an Isaac Newton to see the truth of the matter.

ARGUMENT 3. This is not so much a separate argument, but more a rebuttal to certain objections to Argument 2. Some people might say "shouldn't the earth be falling (hence moving) to the bottom of the universe? Otherwise, what's holding it up?" He replies: the body of the heavens supports it on all sides, and there is no "above" or "below" for the whole universe, anyway, and the earth is itself at the bottom, i.e. the center, where heavy things tend.

ARGUMENT 4. If earth did have a movement down, it would leave behind all other bodies, like animals and people and cars, **because it is so much heavier**, and hence **would fall much faster** than these little things. Ptolemy concludes "And the animals ... would be left hanging in the air." "Merely to conceive such things makes them ridiculous," he concludes.

This is a very funny image, but is it true that heavier things fall faster? Galileo's legendary experiment at the leaning tower of Pisa, whatever the historicity of it, demonstrates that heavier things do not fall faster, or at least not so much as we might think. Obviously a feather will fall more slowly than a bowling ball, but that is because the feather is very bad at dividing the air through which it is falling. If we remove the unequal wind-resistance of the feather and bowling ball, say by dropping them both inside a tube from which the air has been sucked out, they plummet at the same rate. Since most of us don't have a way of sucking all the air out of a tall tube, it might be worth mentioning that there is another much simpler experiment we can perform to show the same thing. A book and a sheet of paper, dropped from the same height above the floor, take very unequal times to fall. The book plummets, while the paper glides or flutters down, significantly slowed in its motion by the resistance of the air. But now place the paper on top of the book (use a piece of paper small enough so no bit of it hangs over the edge of the book), and drop the book. Does the book "leave the paper behind, because it is so much heavier than the paper"? Not at all. The paper sticks right to the book and follows it all the way down!

[5] As for those who say the earth spins on its axis (note some people already thought that back then!), Ptolemy says that this idea might fit with the appearances of the stars, and even be a "simpler" explanation (since you are making just the Earth move, not all the stars and planets), but nevertheless it flies in the face of other things.

(a) To say the earth spins, and not the heavens, means the really heavy bodies, all down here, have a swift motion, while the light bodies, which stay up there or go up there (as fire does), are perfectly still and don't move. Ptolemy and most thinkers in his day thought the heavens must be made of weightless materials—if they were heavy, wouldn't they fall down here to earth? But they don't. They just "stay up there." So they must not have any weight. To make *weightless* things move around in a circle really fast seems more reasonable than to make *the extremely heavy, clumsy, cumbersome earth* move around in a circle really fast, i.e. one giant rotation per day.

(b) The earth's motion would be very swift indeed (around 1000 mph in fact, at its equator). So there should be a 1000 mph wind in the same direction all the time, i.e. contrary to the spin of the earth, i.e. a constant wind from the East (because earth is spinning toward the East). So all clouds should move westward all the time. And anything on wheels (like cars in neutral or shopping carts) should be rolling westward all the time.

Do these arguments prove Ptolemy right? They have a certain plausibility to them. But (a) requires us to believe that the heavenly bodies have no weight. Although they "stay up there" and don't "fall down here" (at least they don't *crash into us*), there could be other explanations for this. Ptolemy is assuming that the "heaviness" of a body is not affected by its distance from the center of the earth—no matter how far away it is, it will still fall toward the center of the earth, and at the same rate. How does he know this? Couldn't it be that each celestial body has its own "down," namely toward its center, and bodies in their vicinity have weight toward them, and not toward other heavenly bodies, or not as much?

And (b) requires us to assume that the air around the earth (our atmosphere), and the various objects standing on the earth, would not themselves share in the spinning motion of the earth. But Ptolemy addresses that next:

[6] If our opponents say the air is also carried around with the motion of the earth, says Ptolemy, then other problems arise. Presumably, my body shares in the motion of the earth only because I am in contact with it, or because it is shoving me along somehow. But then why doesn't the earth move 1000 mph under my feet whenever I jump up in the air? And why don't we *feel* this rapid motion of ours? Why don't we feel the earth moving? When we put objects on a spinning Lazy Susan or a potter's wheel or any rapidly spinning thing, they fly off in all directions, don't they? So why don't we fly off the earth if it is spinning so quickly?

This last is perhaps the strongest argument in favor of Ptolemy's view that the earth is at rest (and hence the heavens, not us, are moving, and so we are at the center of the universe, or thereabouts). If the earth is moving, if it is both spinning on its axis and flying around the sun at enormous speeds, these fantastic motions are for some reason impossible for us to feel by ordinary experience. And anyone who says the earth has such motions will be obliged to explain why we don't feel them.

SCHOLIUM ON SCIENTIFIC REASONING AND MODEL-MAKING

Soon after giving all these arguments for the basic ingredients of his geocentric model of the universe, in Chapter 8 of Book 1 of his *Almagest*, Ptolemy says:

> It will be sufficient for these hypotheses, which have to be assumed for the detailed expositions following them, to have been outlined here in such a summary way, since they will finally be established and confirmed by the agreement of the consequent proofs with the appearances.

There are a number of significant points in this sentence.

(1) First, Ptolemy is acknowledging the inherent weaknesses in the foregoing arguments. He has not really proved that the Earth sits still, that the heavens move and do so on a sphere, that the Earth is at the center of the universe. Almost all he has established with any real decisiveness is that the Earth is sensibly spherical and that it has the ratio of a mere point to the heavens.

(2) Second, Ptolemy is acknowledging the need to adopt *some* "hypothesis," some model or other, at the beginning, prior to having adequate proof of its correspondence to reality. This is necessary because the final "proof" of the model will come from showing how a more detailed version of it can be made to match up beautifully with and "explain" all the specific details of our observations with quantitative exactness. But we cannot develop or even understand that more detailed version of the model without adopting a general notion of the model first.

(3) Third, Ptolemy here seems to be endorsing a certain philosophy of science. He seems to be arguing that, while many models can be made up at the beginning which match the observations in a general way, only a true model will be able to match the observations in all their quantitative specificity. It is like saying that many theories of the crime can match the facts of the case in general, but only the true explanation of the crime can match all of the facts exactly, in all their particularity and detail. It is not so clear that Ptolemy really thinks this, however, since, as we shall see, he will himself propose alternate models of the same particular phenomena which produce the same appearances, and he will make no attempt to decide which one is the "truth." But it is a question worth considering.

Is it true that only the true explanation of a set of observable facts can account for them all? Or is it ever possible, given a certain set of facts, to explain them in quite different ways? Is that always possible?

Since this is our first stab at this question, let's keep things simple—almost childishly simple—by examining this piece of reasoning:

- *If Ptolemy's model corresponded correctly with reality, then X, Y, and Z would be observed* (X, Y, and Z are quantitative consequences of the model which should fall within the realm of what we can observe, for instance one might be that "Venus never gets more than Q degrees away from the Sun").
- *But X, Y, and Z are observed.*
- THEREFORE *Ptolemy's model corresponds correctly with reality.*

Does that conclusion really follow? It does not. In form, it is logically the same as this piece of reasoning:

- If you were in a refrigerator right now, you would feel cold.
- But you do feel cold.
- THEREFORE You are in a refrigerator right now.

Even if both premises are true, the conclusion might still be false. So it does not follow logically from the premises. This way of reasoning (or misreasoning) is called "the fallacy of affirming the consequent."

It is a fallacy, however, only insofar as it is meant to be a necessary argument or "syllogism." But what if we multiply arguments of this type, and what if the "consequents" are things very peculiar and precise, and things which were predicted for the first time *because* someone thought up the theory developed in the "If" part of the first premise? Then the "if" part, which contains some theory or model or hypothesis, starts to look very likely indeed. Imagine a murder investigation is going on, and the theory of the crime, T, specifies who did it, for what motive, when, where, and by what means. That theory leads to certain consequences:

If T is true, then Jones's DNA will be found under the victim's nails.
But Jones's DNA is found under the victim's nails.

If T is true, then Jones's wife must have been unfaithful.
But Jones's wife has been unfaithful.

If T is true, then Jones must not have been seen at the conference he was attending during 4 and 6 pm.
Jones was not seen there during 4 and 6 pm.

If T is true, then Jones must own a cannibal's brain-pick.
But Jones does own a cannibal's brain-pick.

And so on. If we multiply enough checkable consequences like these, and all of them check out, and if the consequences are in some cases quite surprising and unusual, and if they follow necessarily and readily from T, the theory of the crime, then while none of these arguments definitively proves T, taken together they present a very compelling case for the truth of theory T. One might say that T, while not demonstrated with logical necessity, has been established "beyond all reasonable doubt."

Even then, however, we sometimes find we wrongly convict people. And something similar happens in science from time to time. We will return to this over-arching question now and then, especially once we arrive at Copernicus and Kepler.

PTOLEMY

DAY 5

TWO PRIME MOVEMENTS IN THE HEAVENS

In Chapter 8 of Book 1 of his *Almagest*, Ptolemy introduces two more principles or assumptions, namely the two chief movements of the heavens.

THE FIRST MOVEMENT.

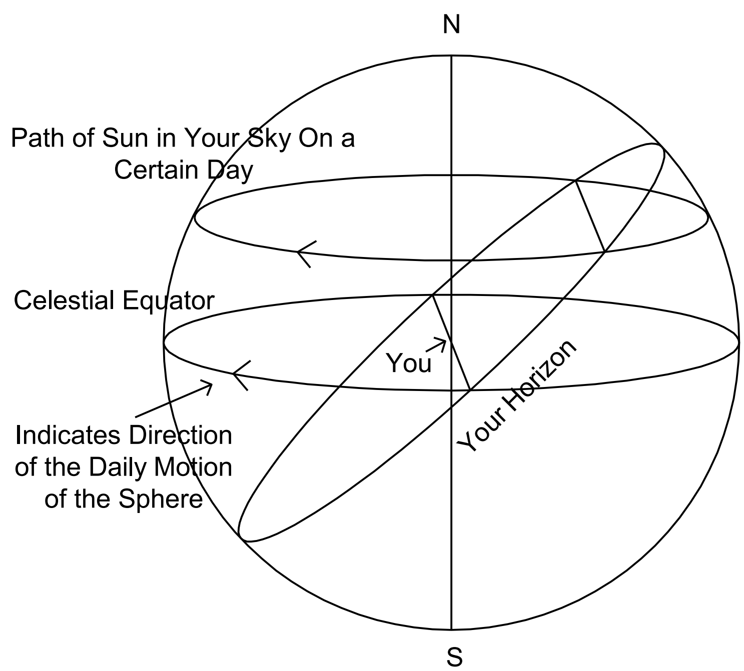
"One is that by which everything moves from East to West, always in the same way and at the same speed, with revolutions in circles parallel to each other and clearly described about the poles of the regularly revolving sphere," says Ptolemy.

This is the "daily" movement of the celestial sphere, and of all things in it. For example, the Sun partakes of this motion, and rises and sets with a motion that is more or less the same as the daily motion of the other stars. It is the same as theirs in that it takes almost the same time (about 24 hours), and in that it is about the same poles (the North and South celestial poles).

Ptolemy continues: "Of these circles the greatest is called the equator, because it alone is always cut exactly in half by the horizon which is a great circle of the sphere, and because everywhere the sun's revolution about it is sensibly equinoctial."

What is he saying here?

As we saw before, a "Great Circle" is a circle on a sphere, the center of which circle is also the center of the sphere. It is called "great" because that is the largest circle you can describe on the sphere. The EQUATOR is one of the great circles on the celestial sphere (in which all the fixed stars are fixed), and it is the only great circle which lies parallel with the E-W motion of the heavens.



It is called the Equator because it is the only circle (in which things move in the daily motion) that is always cut in half by the horizon, regardless of where you live. If you live at the earth's equator, directly beneath the celestial one, then ALL those parallel circles are cut in half by your horizon. But no matter where you live on Earth, your horizon bisects the celestial equator (if you live at the North or South pole, your horizon coincide with the celestial equator). In the accompanying diagram, you can see that the celestial equator is bisected by the horizon of someone standing neither at the pole nor on the earth's equator, but somewhere between. But any other circle traced by the daily rotation of the sphere, for instance the path of the Sun on a particular day, will not be bisected by your horizon (unless your horizon happens to pass through the North and South celestial poles, because you live at the equator).

Another reason he gives for calling this circle the Equator: "because everywhere the sun's revolution about it is sensibly equinoctial."

What does this mean? That when the sun is on the Equator (which happens twice a year, once on its climb North, once in its descent South), its path for that day coincides with the location of the celestial equator in your sky, and so its path is bisected by the horizon no matter where you live, and hence you get equal amounts of daylight and darkness on that day.

THE SECOND MOVEMENT.

Ptolemy now goes on to describe the second principal movement in the heavens: "The other is that according to which the spheres of the stars make certain local motions in the direction opposite to that of the movement just described, and around other poles than those of that first revolution."

The first motion, the "daily" motion, is of the celestial sphere itself, and it is about the North and South celestial poles, and it takes about 24 hours to complete a rotation. This second motion Ptolemy describes is primarily associated with the more particular spheres on which the Sun, the Moon, and the Planets revolve. He imagines the universe is entirely contained within the Sphere of Fixed Stars, or the Celestial Sphere (or "the sphere of the heaven"), but inside that all-containing sphere are *other* spheres, some nested within others, on which the Sun, Moon, and Planets ride about with motions peculiar to themselves. Moreover, these peculiar motions of the Sun, Moon, and Planets, are rotations which are *not* about the North and South celestial poles, but about other poles.

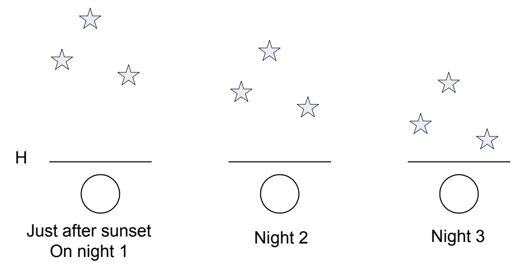
What observations make him say there is a second movement, contrary to the first, which the Sun, Moon, and Planets follow? This takes us back to the exercises of Day 1. Exercises 3 through 6, and the Phenomena associated with them, are the observations which prompted Ptolemy to posit another movement for the Sun in addition to its share in the daily rotation of the celestial sphere.

Let's refresh our memories a bit.

OBSERVATION 1. (A few nights)

A given fixed star sets about 3 minutes, 55.91 seconds *earlier* each night (this is regular year round). So one "sidereal day" (i.e. the time it takes a star to go around the earth) is shorter than a 24 hour day by that much, i.e. one sidereal day = 23h 56m 4.09 seconds.

Watch some constellation setting 1 hour after sunset just where the sun set. It will differ throughout the year.

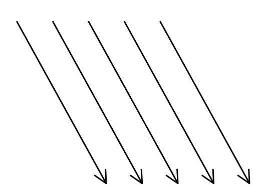


The setting place of a star is basically the same every night (or during the day, if that's when it is in the sky for us). Ignoring the precession of the equinoxes (more on this later), the celestial pole never budges in the sky, and stars are fixed in their positions relative to the celestial poles, and so they rise and set in fixed places on the horizon. (Earth's orbit about the sun is too small to make any difference in where the stars appear to set, and earth's axis remains parallel to itself with only a slight swivel which is completed once every 25,000 years.)

OBSERVATION 2. (A few days)

The sun gets behind the stars in varying amounts (*behind* because the stars are setting almost 4 minutes earlier than 24 hours from their last setting, whereas the sun's time from sunset to sunset is an average of 24 hours) each day, and so it creeps eastward through the sphere of fixed stars, tracing out a full circle in them in one year, called the "ecliptic." (The moon and planets travel roughly along that same circle, i.e. roughly in one plane around the sun; the "ecliptic" is either where the sun appears from our point of view in the stars, or the projection of the plane of earth's orbit.) We have already discussed that, while we cannot "see" where the Sun is among the stars simply by looking at the stars around the Sun (since the Sun's light, when it is not being eclipsed by the Moon, makes the stars impossible to see), we can see which stars rise as the sun sets, and, by checking our star charts, see which fixed stars the Sun must be in.

OBSERVATION 3. (Preferably a few years)



In September, where I live (at about 35° North latitude),
the sun sets further South each evening
& the sun sets about 3 minutes earlier each evening (but this varies throughout the year, i.e. whether it is setting earlier or later, and by how much).

This means that if we tried to insist that the Sun is just moving about the N and S celestial poles, we would have to say it is "jumping" more and more north, and back again, more and more south, "changing lanes" on the celestial sphere from day to day throughout the year. Very irregular!

If, instead, we say that it is creeping slowly along another circle, which is tilted at an angle to the celestial equator, we can say it is just moving uniformly along that circle at the same time as it partakes of the daily motion.

And which way is it going along that other great circle?

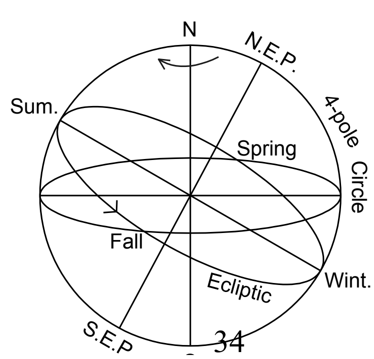
The OPPOSITE way of the general daily motion of the stars, i.e. it is going from W to E along it. How do we know? Because the sun is falling *behind* the stars a little bit each day in motion westward, since the sun takes about 24 hours from sunset to sunset, whereas the same star takes a few minutes less than that to go from setting back to setting. So the star is a few minutes ahead of the sun in moving toward the west.

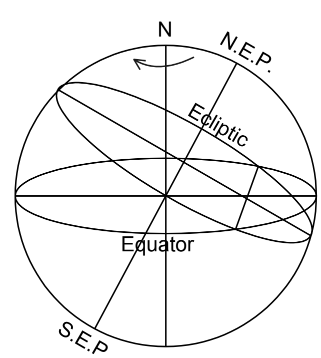
This projection of this circle of the Sun's backward (eastward) motion, onto the sphere of fixed stars, is called the ECLIPTIC. So the ecliptic is NOT the actual orbit or path of the Sun around the Earth, but is the circular line on the sphere of fixed stars to which the Sun's orbit projects from our point of view. (Ptolemy knew that the Sun was *much* closer to us than the "sphere of fixed stars". But we will talk about his understanding of its actual path around Earth later.)

The Moon and other planets also move, roughly, along that path, and in that direction (but they have different periods, or times to complete a single orbit around Earth).

How do we know that the circle on which the sun moves backwards, i.e. the Ecliptic, is a Great Circle?

Because if it were not, it would not be bisected by the





celestial equator, and the sun would never be above and below our horizon for equal amounts of time, i.e. we would never get an equinox. But we do. So it is bisected by the Equator, and so it must share the Equator's center, and be a great circle itself.

So there are FOUR POLES in the heavens, now. There are the North and South Celestial poles, which are the poles of the daily motion. And then there are the North and South Ecliptic poles (the N.E.P. or North Ecliptic Pole, and the S.E.P. or South Ecliptic Pole), which are the poles of the Sun's annual motion. Ptolemy sometimes refers to the great circle which passes through those four poles, what we might call the "4-pole circle."

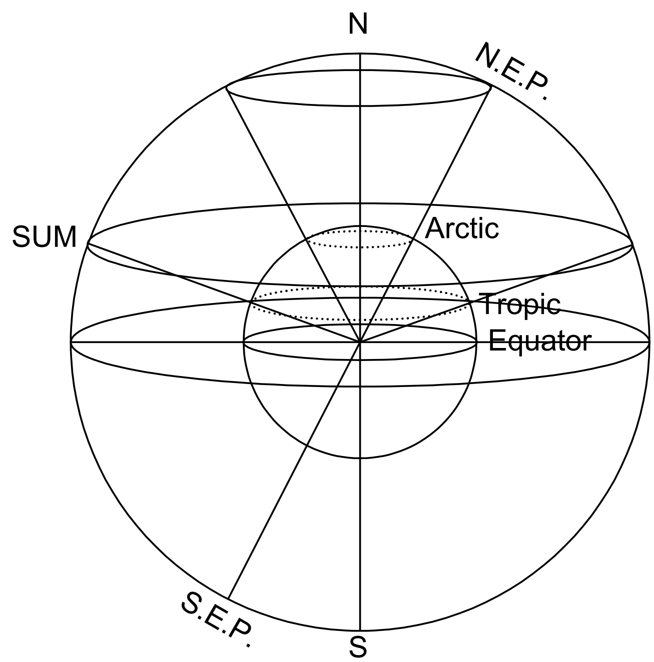
SOME LINES ON EARTH

EQUATOR = the circle on the surface of the earth traced out by the line drawn from earth's center and tracing the celestial equator.

NORTH POLE = the point on earth's surface through which the line joining earth's center to the celestial north pole passes.

TROPIC OF CANCER = the circle on earth's surface generated by joining the center of the earth to the summer tropic (the sun's northmost circle in the heavens) and tracing it out. Similarly for the Tropic of Capricorn in the south.

ARCTIC CIRCLE = the circle on earth's surface generated by joining the center of the earth to the North Ecliptic Pole and tracing out its daily path (and similarly for the Antarctic Circle). It is the first place where you get a 24-hour night in the winter, which happens, in principle, for 1 day of the year at the circle itself, and for more days of the year as one goes north of it.



VOCABULARY

POLE. Point on the celestial sphere that does not move and about which other stars move in a circle. There are 2 poles for each of the 2 prime motions. The N and S poles do not move within the motion of the equator, and the poles of the ecliptic do not move with reference to the motion of the ecliptic.

EQUATOR. Great circle on the celestial sphere perpendicular to the straight line (celestial axis) joining the North and South poles.

HORIZON. Great circle on the celestial sphere perpendicular to you, under your feet (or to a line joining your feet to the center of the earth).

MERIDIAN. Great circle on the celestial sphere passing through the North and South celestial poles and perpendicular to your horizon (or through the point straight over your head). This is called the "meridian", from "mid-day", because when the Sun is on your meridian, it is mid-day for you, and the Sun is halfway through its path over your horizon for the day.

ZENITH. The point straight over your head, i.e. the midpoint of the 180° of the portion of your meridian that is above your horizon. (Opposite to this is your Nadir, the point on the celestial sphere straight below you, on the other side of the earth.)

RIGHT SPHERE.

A view of the heavens where the celestial equator is perpendicular to your horizon, i.e. straight overhead. (i.e. a view of the heavens from earth's equator.)

OBLIQUE SPHERE.

A view of the heavens where the celestial equator is oblique to your horizon, i.e. not straight overhead. (i.e. a view of the heavens from anywhere on earth other than its equator or poles.)

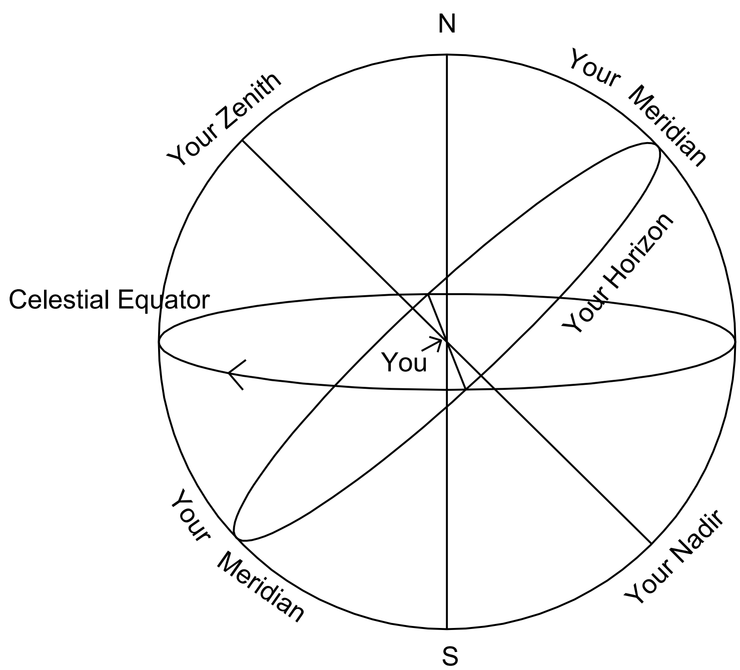
ECLIPTIC. The path of the sun projected onto the sphere of the fixed stars, which is a great circle, completed in an eastward motion of the sun once a year, and which has poles of its own, other than the celestial poles.

EQUINOX. One of two days in the year (one in the fall, another in the spring) when every place on earth has 12 hours of daylight and 12 hours of darkness (except, in a sense, at the poles, where the sun is bisected at the horizon). Or: one of two points where the ecliptic and equator intersect; equinoxes occur whenever the sun is at one of these points in its orbit around us along the ecliptic, so that it is both on the ecliptic and also on the celestial equator (exactly so for a moment, but roughly so for about a day).

SOLSTICE. A day of the year when every place on earth has either maximum sunlight (summer) or minimum sunlight (winter). Or: one of two points on the ecliptic furthest from the equator; solstices occur whenever the sun is at one of these points.

TROPIC. One of the two points on the ecliptic furthest from the equator (another word for a solstice point).

ZODIAC. Band of 12 constellations along the ecliptic spanning about 8° above and 8° below it.



PTOLEMY

DAY 6

THE SEXAGESIMAL SYSTEM AND THE NEED FOR A TABLE OF CHORDS AND ARCS

Before getting into any of the detailed versions of Ptolemy's models for the motions of the Sun and the planets, we need to understand some of his mathematical equipment. To begin with, we need to understand a little bit about his numerical system, and also his need for developing a "table of chords and arcs," which will enable him to find the sizes of lengths and sides in triangles after being given some of the sizes of the other lengths and sides. We will not bother learning how to multiply, divide, and find square roots in Ptolemy's sexagesimal system. That is too much work with too little return, given what we have to do in the course. But it is important to understand what the sexagesimal system is, in order to understand Ptolemy's numbers, and it is useful to be able to turn his numbers into decimal form and our decimal expressions into sexagesimal ones.

DECIMAL SYSTEM.

Ptolemy does not use the decimal system to which we are accustomed, but the much more tedious sexagesimal system. Copernicus and Kepler and Newton also use it to some extent. And it remains in use today, in many applications.

What do we need to know about this system?

(a) It is important to know how to read sexagesimal values, and to be able to translate a sexagesimal value to a decimal one and vice versa, in order to understand Ptolemy's tables and calculations, at least as he presents them.

(b) It is good to know how to add and subtract them (which is just a matter of knowing how to simplify, and how to borrow).

(c) It is *not* important (in my humble opinion) to know how to find square roots in the sexagesimal system, or how to multiply or divide with them. This is tedious, uninteresting, and life is hard enough without it. Simply convert to decimal, do the calculation there, and convert back to sexagesimal.

We are accustomed to using the decimal system of numbers. This means not only that we have 10 basic numerical symbols (0, 1, 2, 3, 4, 5, 6, 7, 8, 9), but also that we signify numbers by making them sums of powers of ten. For example:

$$356 \quad = \quad (3 \times 10^2) + (5 \times 10^1) + (6 \times 10^0).$$

Notice that the place of the digit indicates which power of ten we are multiplying. And if we need to designate fractions of a unit, we do so by cutting it up into equal parts in numbers which are also powers of ten, and then the place of the digit again indicates which power of ten that digit is to be multiplied by:

$$.238 \quad = \quad (2 \times 10^{-1}) + (3 \times 10^{-2}) + (8 \times 10^{-3})$$

Or, putting it a bit differently,

"3.1415" means $3 + \frac{1}{10^1} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4}$

and generally

"a.bcde ..." means $\frac{a}{1} + \frac{b}{10^1} + \frac{c}{10^2} + \frac{d}{10^3} + \frac{e}{10^4} \cdots$

SEXAGESIMAL SYSTEM.

This way of representing numbers uses 60 as a base instead of 10. Instead of saying how many tenths (or hundredths etc.) we have, we say how many sixtieths (or thirty-six-hundredths etc.) we have.

And usually we only take things to the second sexagesimal place after the whole number, i.e. to the 3600ths place. The whole number place is signified by a superscript of whatever unit we are using (e.g. H for Hours, or ° for Degrees) and the "firsts" place by one superscript minute-mark, and the "seconds" place by two superscript minute-marks, thus:

37^{H} 14' 53''

which would be read "37 Hours, 14 Minutes, 53 Seconds."

If we are dividing arcs of a circle, or angles, then we write:

37° 14' 53''

which we read "37 degrees, 14 arc-minutes, 53 arc-seconds."

We still divide time sexagesimally, i.e. we divide 1 hour not into "10 minutes" but into "60 minutes", and one minute not into "10 seconds" but into "60 seconds" (3600ths of an hour).

So generally

a b' c'' d''' ... means $\frac{a}{1} + \frac{b}{60^1} + \frac{c}{60^2} + \frac{d}{60^3} + \frac{e}{60^4} \cdots$

FROM SEXAGESIMAL TO DECIMAL

To convert from sexagesimal to decimal is easy:

$24^{\circ}\ 31'\ 12'' \quad = \quad 24 + 31/60 + 12/3600 \quad = \quad 24.52^{\circ}$

FROM DECIMAL TO SEXAGESIMAL

This is more painful, but still not too bad. Suppose you want to translate the decimal expression 35.398° into sexagesimal form.

$$35.398 = 35 + 398/1000$$

Now we want to turn that fraction into a fraction over 3600 (if we are going to take it out only to the "seconds" place). So we multiply 398 by 3.6 (since that is what we want to multiply the denominator by), giving us 1432.8, which we can round up to 1433. Now we have

$$35.398 = 35 + 1433/3600$$

But just as every 60 seconds is a minute, so too every 60 of our 3600ths is a "first" or "minute" in sexagesimal notation. So how many 60s are there in 1433?

$$1433 \div 60 = 23.88333...$$

So there are basically 23 sixties in there. But 23 × 60 = 1380, and

$$1433 - 1380 = 53$$

leaving us with

$$35.398 = 35 + 23/60 + 53/3600$$

i.e. $35.398 = 35^{\circ}\ 23'\ 53''$

EXERCISES

Turn the following decimal expressions into "hours, minutes, seconds" :

23.468 hours
5.203 hours
14.777 hours

THE NEED FOR A TABLE OF CHORDS AND ARCS
(In other words: The Need for Trigonometry)

In Chapter 10 of Book 1 of his *Almagest*, Ptolemy begins preparing the mathematical equipment he will need in order to develop the detailed versions of his models of solar, lunar, and planetary motions. He has already outlined the whole universe, but coarsely, with no numerical values. We don't have any sense of the proportions of things, or the speeds of things, and so on. Remember, the universe, according to Ptolemy, consists of nested spheres of various sizes moving in various ways. What are the sizes of those spheres? What is the order of them? Is the Sun closer to us, or Venus? How many spheres are responsible for all the movements we see in the planet Venus? We want a mathematically crisp picture of all this. And we cannot get that without the help of trigonometry and a table of chords and arcs.

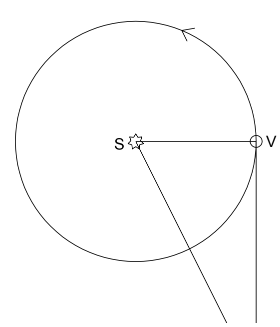
Ptolemy is very brief in explaining how a table of chords and arcs (which corresponds closely to, but is not identical to, a modern table of sines or cosines) is useful for this end. It might be worth understanding this better before jumping into the details of deriving such a table, especially for those not familiar with the usefulness of trigonometry.

The general reason we need such a table is to enable us to solve triangles. A table of chords and arcs will take arcs of a circle of varying lengths (e.g. 1°, 2°, 3°, etc.) and say how long the chords are subtending each of these arcs (compared, say, to the diameter of the circle), and hence also, conversely, it will give us chords in the circle of varying lengths, and tell us how big an arc it cuts off or subtends.

Such a table (as we will see) will enable us to "solve triangles." That is, given sufficient numerical information about certain sides and angles in a triangle, we will be able to say what the values are for all the remaining angles and sides. Hence "**trigonometry**" or "triangle-measurement."

And why do we want to be able to "solve triangles"? To be able to solve triangles is extremely useful in countless ways, not only in pure and exact mathematics, but also in order to get very accurate values for things we cannot measure directly.

Let's see how that can be so.



EXAMPLE 1: ASTRONOMICAL EXAMPLE WITH A RIGHT TRIANGLE.

We want to know the shape of the universe, e.g. the ratios of the various orbits of the planets around the sun (to keep it simple, let's be

Copernican for now, and assume the orbits are all perfect circles with the sun right at the center of all).

We'll start with trying to find the ratio of Venus's orbit to our own. ES is the radius of our orbit, and VS is the radius of Venus's orbit.

How can we find the relative sizes of these two orbital circles? We cannot measure their radii directly!

In fact, we pretty much NEVER OBSERVE DISTANCES directly in astronomy, but *only angles*. This fact alone is a major reason why we need a way of translating angles into corresponding lengths in the sides of triangles.

We orbit S, the Sun, and our orbital radius is ES.

Venus, or V, orbits S, too.

Since Venus's orbit is inside ours, Venus will never seem to get further from S than by some angle SEV, where EV is tangent to the orbit of Venus. So when V is as far away from the Sun as it can get, i.e. when ∠SEV is as large as it ever gets, a "greatest elongation" of Venus is taking place, and then we know EV (our line of sight to the planet) is tangent to the orbit, and hence we know that ∠EVS is 90°. But we can also measure ∠SEV by direct observation. Hence all the angles of △SEV are known in degrees. Hence all the ratios of the sides are determined—there is no flexibility there.

If we had a way of knowing, from these three angles, what the ratios of the sides must be, we could quantify the ratio SE : SV, which would be the ratio of the orbital radii of Earth and Venus. To do that, however, we need a way to solve triangles, and a way to associate certain angles with certain lengths of sides in triangles.

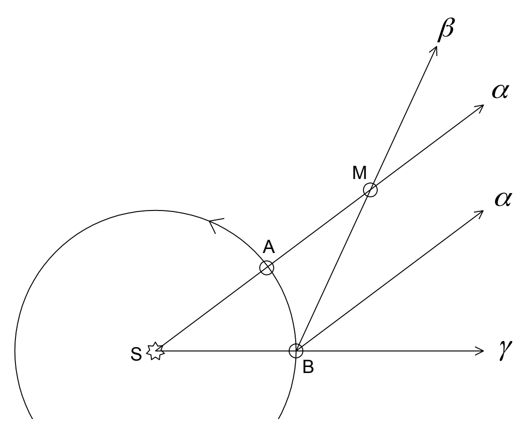
EXAMPLE 2: ASTRONOMICAL EXAMPLE WITH A NON-RIGHT TRIANGLE.

Suppose we want to find the ratio of Mars's orbit to our own.

We orbit S, the Sun, going from B to A.

Mars, or M, orbits S, too.

In the present example, suppose Earth is first at A, exactly between the Sun and Mars, so that SAM is a straight



line (and Mars is rising exactly as the Sun is setting). We observe Mars against the star α in the zodiac.

Suppose we also know that Mars moves at a uniform rate on its circular orbit, and so we know exactly how long it will take Mars to come around again to position M, so at a certain time we will know that $SM\alpha$ are again in a straight line. We note the passage of time starting from when we observe Mars at M (when we are at A); when the time required for Mars to get back to the very same spot on its orbit (its "period") has elapsed, we know that Mars is back at M.

Hence we know Mars is really back at M, even though we are now at B in our orbit, and it does not look to us, from our new vantage point, like Mars is back in the same spot we saw it in before. Now Mars looks to us as though it is superimposed on star β in the zodiac.

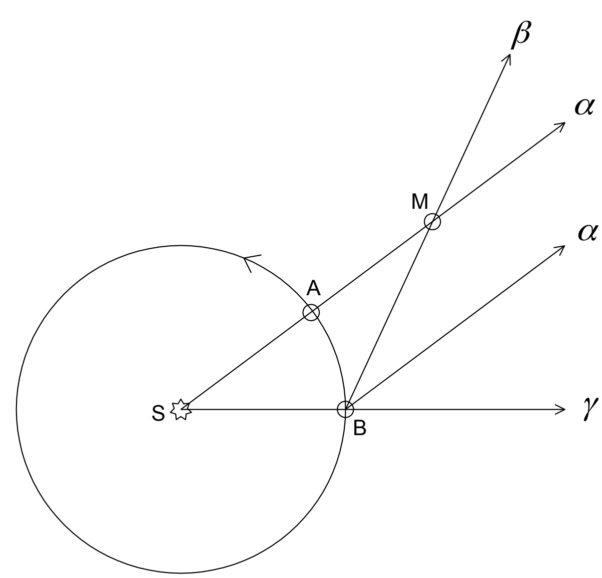
And since our *orbit* (not just our planet!) is as a point to the heavens, therefore $B\alpha$ is pretty much exactly parallel to $A\alpha$,

hence $\angle\beta B\alpha = \angle\beta M\alpha$
but $\angle\beta B\alpha$ is simply *observed and measured*
so $\angle\beta M\alpha$ is now known in degrees.

Thus $\angle BMA$ is now known in degrees,
(since it is equal to $\angle\beta M\alpha$, since they are vertical angles).
Or, in other words, $\angle BMS$ is known in degrees.

But $\angle SBM$ is also known in degrees, since we know that star γ is rising just as the Sun is setting, so that $SB\gamma$ is a straight line, and we can measure the angle γBM directly (with some instrument like a set of graduated circles, or something more sophisticated like a sextant). But this angle is the supplement of $\angle SBM$. Hence $\angle SBM$ is known in degrees.

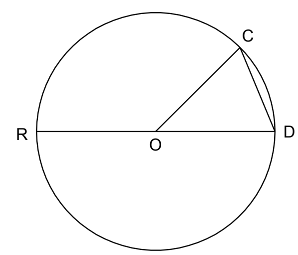
So now all the angles in $\triangle SBM$ are known. Can we, now, assign numerical values to the ratios of the sides of that triangle? Suppose we called SB, the radius of Earth's orbit, "1". Then what would we have to call SM, the radius of Mars's orbit? The triangle SBM is entirely decided in shape. If only we had a way to solve for its other sides, given a numerical value for one of the sides and given all the angles! Hence the need for a table which will enable us to say how big a side subtends a given angle in a triangle, or how big an angle subtends a given side in a triangle.



NOTE ON THE EXACTNESS OF TRIGONOMETRY.

Trigonometry is sometimes mistakenly thought to be an inexact, good-enough system. That is not so. It is perfectly exact and mathematically rigorous, in itself. But we often use its laws to determine values for sides or angles only to a given degree of precision, due to limitations (say) in the precision of our original values, since they resulted from physical measurement, or else due to limitations to our own ability to carry out calculations to infinite decimal (or sexagesimal) places.

Very well, then, our next business will be to develop the fundamentals of trigonometry. This will consist in building up a "Table of Chords and Arcs," which corresponds to a modern table of Sines and Cosines (as we shall see later). Imagine a circle of center O, with diameter ROD. Now suppose I laid off a chord inside the circle, CD, and told you exactly how big it was compared to the diameter, say one third the length of the diameter. Could you tell me how big the angle DOC is in degrees? You can if you have a table of chords and arcs! Again, suppose I did the reverse, and told you exactly how big the angle DOC is in degrees—say it is 34.592°. Can you tell me how long the chord CD must be in terms of the diameter? You can if you have a table of chords and arcs! And you can see that if we can answer questions like this about the triangle DOC, we will be well on our way to being able to calculate the values for all the sides and angles of any triangles, given sufficient information about them.

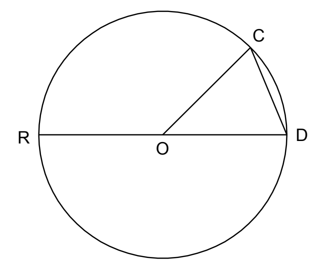


PTOLEMY

DAY 7

BUILDING A TABLE OF CHORDS AND ARCS: PART 1

(1) WHY 360 AND 120?



Ptolemy divides his standard circle's diameter, DR, into 120 parts, in keeping with his sexagesimal ways—this way the radius of the circle is 60 of those 120^{th}-parts of the diameter, and each of those is called a "part of the diameter's 120". And he divides the arc of the circle (and hence the angles subtending them at the center of the circle) into 360 equal arcs, each one called a "degree."

As cumbersome as this is for us, there were some advantages to this for Ptolemy, who did not have decimal calculators at his disposal.

(a) For one thing, $120 = 5! = 1\times2\times3\times4\times5$, so 120 has lots of integral factors. 360 is triple that, so it also has lots of factors. This makes for more whole numbers of degrees when we cut the circumference of the circle into some whole number of parts; e.g. "half a circle," "a third of a circle," "a quarter of a circle," "a fifth of a circle," will all have whole number degree-values, as will a tenth, a twelfth, a fifteenth, and so on.

(b) 360 : 120 = 3 : 1, a rough approximation of pi. In other words, one "degree" of arc is *almost* equal to one "part" of the diameter.

(c) With this system, the equilateral triangle on the radius has 3 angles of 60° and 3 sides of 60 parts. That's rather pretty.

(d) There are 365 days in a year; so now we have the Sun moving about 1° a day on its circle (just a little less, as we shall see).

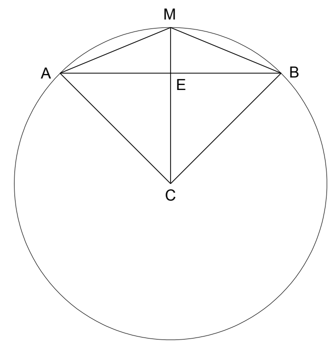
(2) CHORDS ARE NOT AS ARCS.

To see the need for this table of chords and arcs, we need to see that chords do not have the same ratios as the arcs they subtend. We easily fall into the mistake of thinking that any two chords in this circle will have to each other the same ratio as the arcs of the circle they cut off. If you are tempted to think this, banish it from your thought! It is not true. And it is easy to forget that this is not true, even for some veterans of trigonometry. One must burn this annoying fact into memory. If chords were as arcs, then it would be much easier to build up the Table! For instance, since the chord of 180° is 120 parts, it would follow that the chord of 90° must be 60 parts, and so on proportionally.

Alas, it is not so. A very quick proof of this: Draw a square inscribed in our circle. Obviously the chord of 90° is the side of that square, and the chord of 180°, the diameter, is the diagonal. But the diagonal is not double the side, even though the arc is double the arc.

Or draw a regular HEXAGON in a circle, letting AB, BC, CD be three consecutive sides. Then AD is the diameter of the circle, and is double AB. But the arc AD (i.e. ABCD) is not double the arc AB, but triple it!

Again,



If	∠ACM = ∠BCM,	
and so arc AM = arc MB		
so	arc AM = ½ arc AMB	
is	chord AM = ½ chord AB ?	
No.	AE = ½ AB	
and	AM > AE	(hypotenuse)

(3) PLAN FOR BUILDING THE TABLE:

So we must find other inroads into filling out the entries on our table.

Plan of attack:

[1] FIND CHORDS OF 36° & 72°.
= items (4) – (5) below.

[2] FIND CHORDS OF 60°, 90°, 120°
and note that *the chord of the supplement to an angle whose chord is given, is given.*
= item (6).

[3] FIND CHORDS OF 144°, 108°.
= item (7).

[4] PROVE THAT CHORDS OF ARCS WHICH ARE "DIFFERENCES OF ARCS WITH GIVEN CHORDS" ARE GIVEN.
= items (8) – (9)

[5] PROVE THAT CHORDS OF ARCS WHICH ARE "HALVES OF ARCS WITH GIVEN CHORDS" ARE GIVEN.
= item (10)

[6] PROVE THAT CHORDS OF ARCS WHICH ARE "SUMS OF ARCS WITH GIVEN CHORDS" ARE GIVEN.
= item (12)

[7] FIND CHORDS OF 1° AND 1½ °.
= items (13) – (15)

[8] INTERPOLATION OF "SIXTIETHS"
= item (16)

[9] RELATION OF TABLE OF CHORDS TO TABLE OF SINES
= item (17)

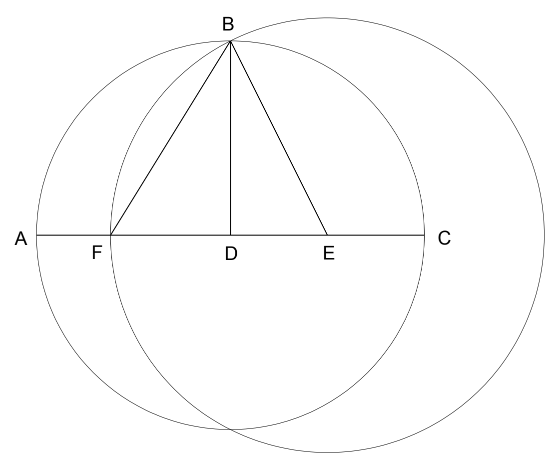
NOTE:

36°, 72°, 60°, 90° are all found directly.
120°, 144°, 108° are found as supplements.
12° is found by "subtraction", i.e. by [4] above.
6°, 3°, 1½°, ¾° are found by "bisection", i.e. by [5] above.
All multiples of 1½° are found by "addition", i.e. by [6] above.
1° and ½° are found by approximation.

We will cover Steps [1] through [6] today, i.e. items (4) through (12), and we will cover Steps [7] through [9], i.e. items (13) through (17), in Day 8.

(4) PRELIMINARY TO FINDING THE CHORDS OF 36° & 72°.

First, Ptolemy shows that if we have a circle of diameter ADC, perpendicular radius DB, and bisect radius DC at E, and join EB, and draw a circle with center E, radius EB, cutting radius DA at F, and join FB, then

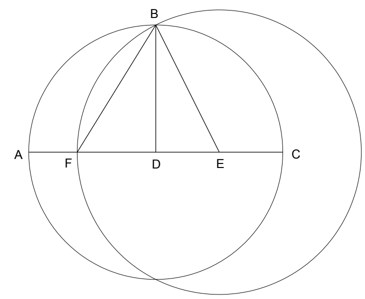


FB = side of regular inscribed pentagon
FD = side of regular inscribed decagon

He uses Euclid's *Elements*, Book 13 Proposition 10, for this.

So now we know that FB is the chord of 72°, and FD is the chord of 36°. This does not give us a numerical value for them yet, in terms of the 120 parts of diameter AC, but it will enable us to do that in the next step.

(5) FINDING THE CHORDS OF 36° & 72°.



Now let's find numerical values, as precise as we like, for the chords of 36° and 72°:

	AC = 120	[given]
so	ED = ¼ AC = 30	
and	DB = ½ AC = 60	
so	BE = $\sqrt{ED^2 + DB^2} = \sqrt{900 + 3600} = \sqrt{4500} = 67.08203932...$	
so	EF = BE = 67.08203932...	
so	FD = EF – ED = 67.08203932... – 30 = 37.08203932...	
i.e.	Chord 36° = 37.08203932...	

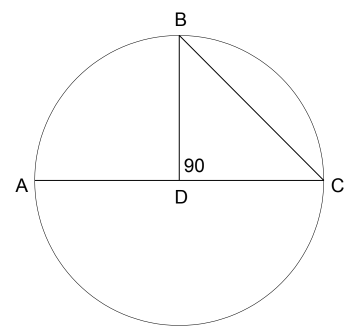
so	$FD^2 = 1375.07764...$
but	$DB^2 = 3600$
so	$BF^2 = FD^2 + DB^2 = 1375.07764... + 3600 = 4975.07764...$
so	BF = 70.53423027...
so	Chord 72° = 70.53423027...

NOTES:

(a) These are decimal values, but are easily translated into sexagesimal values matching those on Ptolemy's Table.

(b) Ptolemy speaks of "getting" or "finding" chords, or of chords being "given." What he means is to *find a way of determining a numerical value for their lengths, in units of one-hundred-twentieth parts of the diameter, to any desired degree of precision.* We do this by beginning with chords whose exact values are known for geometrical reasons, then by showing how the sought chord is the result of a known operation on the known chords.

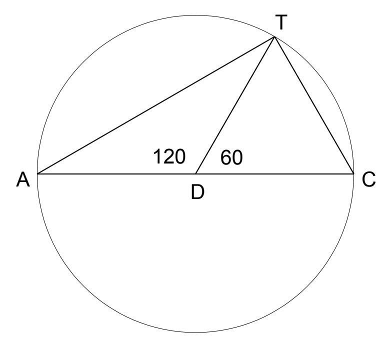
(6) FINDING THE CHORDS OF 60°, 90°, 120°.



Chord 60° = Radius = 60

so $BC^2 = 2DC^2 = 2 \times 3600 = 7200$
so BC = 84.85281374...
so Chord 90° = 84.85281374...

(NOTE: I will stop putting the ellipsis in (...) just for simplicity. I will just truncate the expressions at an arbitrary place.)



If we now let ∠ADT = 120°, so that ∠CDT must be 60°, we know:

$AT^2 = AC^2 - CT^2 = 120^2 - 60^2 = 14400 - 3600 = 10800$
so $AT = \sqrt{10800} = 103.9230458$
so Chord 120° = 103.9230458

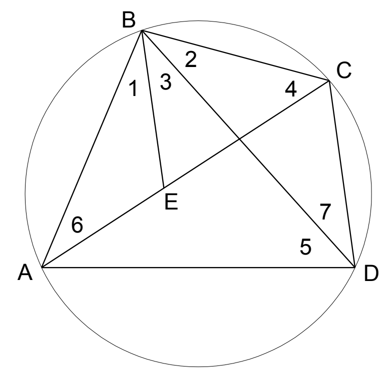
NOTE: WE CAN FIND CHORDS OF SUPPLEMENTS. We can now find the chord for any angle which is the supplement of an angle whose chord is known. In the example above, we knew CT, the chord of 60°, and we knew AC, the diameter. That, together with the Pythagorean Theorem, was all we needed in order to compute AT, the chord of the supplement of 60°. There was nothing special about 120° and 60°. So now, if we know the chord of any angle, we will also be able to compute the value of the chord of its supplement.

(7) FINDING THE CHORDS OF 144° & 108°.

So using the very same technique, we can find the chords of 144° and 108°, since these are the supplements of 36° and 72° respectively, and the chords of those arcs are known.

(8) LEMMA (FOR THE UPCOMING PROOF THAT CHORDS OF ARCS WHICH ARE "DIFFERENCES OF ARCS WITH GIVEN CHORDS" ARE GIVEN).

This may be called the "Diagonal Cross-Product Theorem."



The Theorem states that if ABCD is a cyclic quadrilateral (i.e. a quadrilateral inscribed in a circle), then

AC·BD = AB·CD + BC·AD

or, put verbally: The rectangle contained by the diagonals is equal to the sums of the rectangles contained by the pairs of opposite sides.

Again, since this is just a matter of going through the steps, we will assume it is true and use it, but not bother proving it together.

(9) PROOF THAT CHORDS OF ARCS WHICH ARE "DIFFERENCES OF ARCS WITH GIVEN CHORDS" ARE GIVEN (i.e. calculable).

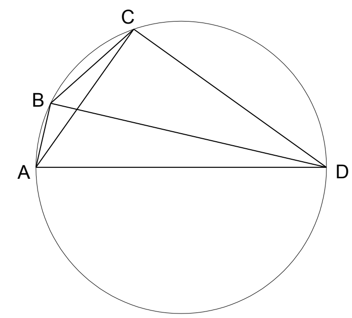
Given: arcs AB & AC in degrees; chords AB & AC in 120^{th} parts of diameter AD
Prove: chord BC is also given in 120^{th} parts of AD (i.e. can be calculated)

Since AB & AD are given, thus BD is given [Euclid 1.47]

Since AC & AD are given, thus CD is given [1.47]

But AC·BD = AB·CD + BC·AD [by the cross-product Lemma]
and all terms in that equation are given except for BC.
Hence BC is also given.

Q.E.D.



NOTE: We are not just thinking of AC.BD (for example) as a rectangle, but as a product of two numbers. There's plenty of philosophically discussible matter there.

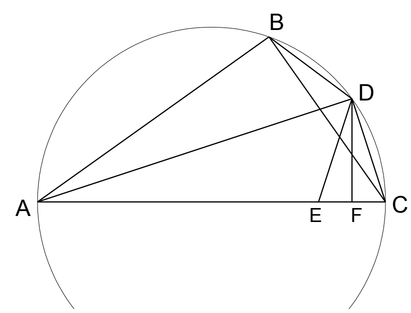
EXAMPLE of the use of this Theorem for our Table:

If arc AC = 72°
and arc AB = 60°
then arc BC = 12°. And since the chords of 72° and 60° are already given, it follows by this Theorem that the chord of 12° is also given now, or calculable to any degree of accuracy we please.

(10) PROOF THAT CHORDS OF ARCS WHICH ARE "HALVES OF ARCS WITH GIVEN CHORDS" ARE GIVEN.

Ptolemy next shows that if we know the chord of a known arc, then we can also calculate the value of the chord of half that arc. That will help fill in a whole lot of entries on the table!

If we know arc CB in degrees (and its midpoint is D), and chord CB in 120th parts of diameter AC, then we will also be able to calculate the value of chord CD in those units.



Given:	arc CB in degrees chord CB in 120th parts of the diameter AC arc BD = arc DC
Prove:	CD is given (i.e. the chord of half the given arc CB)
Make:	AE = AB DF perpendicular to AC

	AB = AE	
	AD common	
	∠BAD = ∠EAD	[Euclid 3.27; they stand on equal arcs]
so	△BAD ≅ △EAD	
so	BD = DE	
i.e.	CD = DE	
so	△DEF ≅ △FCD	
so	CF = EF	
i.e.	CF = ½ EC	
so	CF = ½ [AC – AE]	
	CF = ½ [AC – AB]	
so	CF is given	[since AC is given, AB is supplement of given]

Now	AC : CD = CD : CF	[△ADC similar to △DCF; 6.8]
so	$AC \cdot CF = CD^2$	
so	CD^2 is given	[AC & CF are given]
so	CD is given	

Q.E.D.

EXAMPLES:

Since we had the chord of 12°,
now we have the chords of 6°, 3°, 1½ °, ¾ °

(11) ARCS WHOSE CHORDS ARE NOW GIVEN:

Direct Geometry:		Supplements:
36°	(side of decagon)	144° = 180° – 36°
60°	(radius, side of hexagon)	120° = 180° – 60°
72°	(side of pentagon)	108° = 180° – 72°
90°	(side of square)	
180°	(diameter)	

Subtraction: Supplements:

24° = 60 – 36	174° = 180 – 6
12° = 72 – 60	168° = 180 – 12
18° = 90 – 72	162° = 180 – 18
30° = 90 – 60	156° = 180 – 24
54° = 90 – 36	150° = 180 – 30
48° = 108 – 60	132° = 180 – 48
84° = 120 – 36	126° = 180 – 54
6° = 18 – 12	96° = 180 – 84

Bisection:	Supplements:
42° = 84 ÷ 2	138° = 180 – 42
66° = 132 ÷ 2	114° = 180 – 66
78° = 156 ÷ 2	102° = 180 – 78

Now we have all multiples of 6°

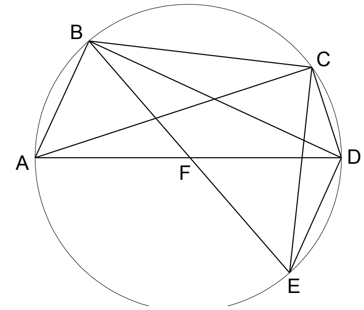
Bisection:	Supplements:
3° = 6 ÷ 2	177° = 180 – 3
9° = 18 ÷ 2	171° = 180 – 9
15° = 30 ÷ 2	165° = 180 – 15
21° = 42 ÷ 2	159° = 180 – 21
27° = 54 ÷ 2	153° = 180 – 27
33° = 66 ÷ 2	147° = 180 – 33
39° = 78 ÷ 2	141° = 180 – 39
45° = 90 ÷ 2	135° = 180 – 45
51° = 102 ÷ 2	129° = 180 – 51
57° = 114 ÷ 2	123° = 180 – 57
63° = 126 ÷ 2	117° = 180 – 63
69° = 138 ÷ 2	111° = 180 – 69
75° = 150 ÷ 2	105° = 180 – 75
81° = 162 ÷ 2	99° = 180 – 81
87° = 174 ÷ 2	93° = 180 – 87

Now we have all multiples of 3°, and by Bisection:
1½ ° = 3 ÷ 2

(12) CHORDS OF ARCS WHICH ARE SUMS OF ARCS WITH GIVEN CHORDS ARE GIVEN

(this is like "addition")

Ptolemy adds this Theorem that will help to fill in new entries on our Table:



Given: Arcs AB and BC are given in degrees
Chords AB and BC are given in diameter-parts

Prove: AC is also given in diameter-parts

Make:	Diameters BFE and AFD	
	CE is given	[supplement of BC]
	BD is given	[supplement of AB]
	DE is given	[supplement of BD]
But	BD·CE = BC·DE + CD·BE	[quadr. multiplication Lemma, item (11) above]
so	CD is given	[all other terms given in there]
so	AC is given	[supplement of CD]

Q.E.D.

APPLICATION:

We can add 1½ ° to itself thus, getting 1 out of every 3 terms on the table. Now we have

1½ ° 3° 4½ ° 6° etc.

NOTE: The use of "given" here means *can be calculated to whatever degree of accuracy you please*, e.g. to as many decimal or sexagesimal places you want. Sometimes we need numerical exercises to get the gist of this. For instance, we have seen that

Chord 36° = 37.08203932...

and Chord 1½ ° = 1.570833333...

so now, using this Theorem, see if you can determine the numerical value for Chord 37½ ° .

PTOLEMY

DAY 8

BUILDING A TABLE OF CHORDS AND ARCS: PART 2

It remains for us to find the chords of 1° and of ½°, which, together with all our other techniques, and with a method (described near the end of today's discussion) for interpolating good values for arcs less than ½°, will give us a fairly complete table of chords and arcs.

(13) PRELIMINARIES TO FINDING THE CHORDS OF 1° & ½° .

(a) It is not possible to trisect an angle of 60° (or to construct the arc of 20°) using only circles and straight lines and Euclid's postulates for plane geometry. (That impossibility is strictly provable, but we will not get into this here.)

(b) Therefore we have no simple geometrical construction to give us calculation-insight into the chord for 20°. That is why it is still missing from our table.

(c) But 21° is already on our Table (see Day 7).

(d) Since we can use simple geometry to get the chord for 21°, but not for 20°, we also cannot use simple geometry to get the chord for 1°, since this is their difference, and we know how to find the chords of arcs which are differences of arcs with known chords. (See Day 7.)

(e) And since we can't use simple geometric techniques to get the chord of 1°, neither can we use them to get the chord of ½°, since if we could get ½° we could also get the chord of its double, 1°, since we know how to find the chords of arcs which are sums of arcs with known chords. (See Day 7.)

(f) But we really need the chords of 1° and ½°! The Table will be woefully incomplete without these and all the other increments of these which are still missing (like 11°, 13°, 17°, and all others which differ by 1° from those already on the Table).

(g) So we need a special way of approaching these, some way of getting a very good APPROXIMATION. This is the one place where Ptolemy does not enable us to calculate to an arbitrary degree of precision. Still, he will get us almost to the nearest 3600^{th} of a 120^{th} part of a diameter, which is more than good enough for naked-eye astronomy, and is as far out as he takes the other entries on the table anyway, i.e. to the third sexagesimal place. (Note: More recent developments in trigonometry enable us to determine the chord of 1°, or the chord of any given arc, to any degree of precision. But we are interested in the beginnings of things right now, so we will follow Ptolemy's way.)

(14) LEMMA: Greater Chord : Lesser Chord < Greater Arc : Lesser Arc.

In order to get a good value for the chords of 1° and ½°, Ptolemy first gives us a perfectly rigorous mathematical demonstration, showing that:

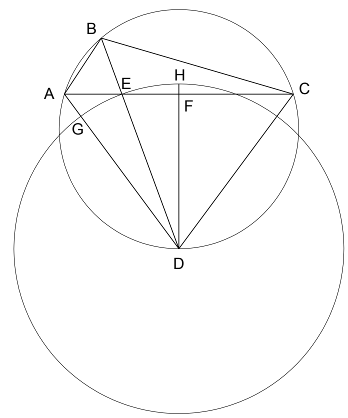
If arc BC > arc AB

then BC : AB < arc BC : arc AB

So not only is it the case that "unequal arcs are NOT as their chords," we can now specify that the greater chord has a lesser ratio to the lesser one than the corresponding arc has to the corresponding arc.

The proof is a little challenging, but here it is:

GREATER CHORD : LESSER CHORD < GREATER ARC : LESSER ARC



Given: arc BC > arc AB [hence BC > AB]

Prove: BC : AB < arc BC : arc AB

Make: DB bisect ∠ABC
DF perpendicular to AC
ED = DH = DG

Now	CD = AD	[arc AD = arc DC, by bisection]
and	CE > AE	[CE : AE = CB : BA, Euc. 6.3]
and	DE > DF	[∠DFE is right, so DE is hypot in ΔDEF]
and	AD > DE	[∠DEA obtuse]
so	G is on AD and H is beyond DF.	

thus	Sector DEH > △DEG[whole > part]
and	△DEA > Sector DEG[whole > part]

so	△DEF : △DEA < △DEF : Sect DEG[1]
so	△DEF : △DEA < Sect DEH : Sect DEG[2]

so	EF : AE < ∠FDE : ∠EDA[3]
so	EF + AE : AE < ∠FDE + ∠EDA : ∠EDA[4]
i.e.	AF : AE < ∠FDA : ∠EDA
so	2AF : AE < 2∠FDA : ∠EDA
i.e.	AC : AE < ∠CDA : ∠EDA
so	AC – AE : AE < ∠CDA – ∠EDA : ∠EDA[5]

[1] Since the same has a lesser ratio to the greater of 2 magnitudes, Euc. 5.8.
[2] Since Sect DEH is greater than △DEF, which just makes it worse, i.e. makes the ratio on the right still larger.
[3] The first ratio is the same as that of the triangles, by Euc. 6.1, while the second is the same as the Sectors, by Euc. 6.33.
[4] "Componendo," Euc. 5.18.

i.e. CE : AE < ∠CDB : ∠BDA
so BC : AB < arc BC : arc AB[6]

Q.E.D.

NOTE: It is not necessary for AC to be a diameter, but DF has to be one.

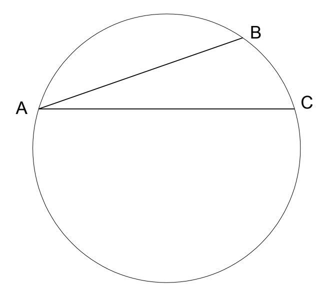
(15) FINDING THE CHORDS OF 1° & ½° .

By our earlier techniques, we were able to get the value of Chord for 3°.
Thus, by our "bisection" theorem, we also have the value of Chord for 1½°.
Thus, by our "bisection" theorem, we also have the value of Chord for ¾ °.

In the exaggerated accompanying diagram,

arc AB = ¾ °
arc AC = 1° = 4/4° = arc AB + ¼ ° = arc AB + ⅓ arc AB = 1⅓ arc AB

Now	AC : AB < arc AC : arc AB	[By the previous Lemma]
so	AC : AB < 1⅓ arc AB : arc AB	
so	AC < 1⅓ AB	[since 1⅓ AB : AB = 1⅓ arc AB : arc AB]
so	AC < 1⅓ of (0^P 47' 8'')	[Chord of ¾°, to our degree of precision]
so	AC < 1^P 2' 50''	
so	Chord of 1° < 1^P 2' 50''	[This is actually true]



Again, let

arc AB = 1°
arc AC = 1½ ° = 1½ arc AB

Now	AC : AB < arc AC : arc AB	[Lemma]
so	AC : AB < 1½ arc AB : arc AB	
so	AC < 1½ AB	[since 1½ AB : AB = 1½ arc AB : arc AB]
i.e.	1^P 34' 15'' < 1½ AB	[Chord of 1½°, to our precision]
so	1^P 2' 50'' < AB	[taking ⅔ of each side]
i.e.	1^P 2' 50'' < Chord of 1° [7]	

---

[5] "Separando," Euc. 5.17.
[6] Euc. 6.33 again.
[7] This is strictly false, but would be true if the chord of 1 ½ ° were EXACTLY equal to 1^P 34' 15''; thus this conclusion is no more false than our approximation of the chord of 1 ½°, which means it is good enough an approximation for us.

So $1^P\ 2'\ 50'' <$ Chord of $1° < 1^P\ 2'\ 50''$
i.e. Chord of $1° = 1^P\ 2'\ 50''$ [i.e. good approximation]

NOTE:

By the chord-of-half-the-arc theorem, chord of ½° is given now, too.
By the addition theorem, every chord of multiples of ½° is given, too.
So the whole table is now complete.

(16) "INTERPOLATION" and the "SIXTIETHS COLUMN".

The table of chords so far contains only ½° intervals.
What if we need to find the chord of 22⅔° ?

Ptolemy gives us a technique for finding the value of the chord of any arc which is in a whole number of minutes, or "sixtieths" of a degree.

Now ⅔ is equal to 40/60, so that is a whole number of minutes (note again the convenience of using 60; almost any ordinary fraction of a degree will be a whole number of sixtieths of it). Here is what we do:

We find the chords on our table between which the chord of 22⅔° must lie, namely the chords of 22½° and 23°. Since there are 30 minutes (i.e. ½°) of arc between these arcs, there are 30 different chords for each of those minutes added to the 22½°. Since ½° is fairly small, we assume that these progressive chords increase (roughly, but close enough for our purposes) by the same increment. Hence we take the difference of the chords for 22½° and 23°, and *divide this difference into 30 equal parts*. Then, for every number of additional minutes beyond 22½°, we add one of these increments to the chord of 22½°, and get a reasonably precise value for the chord of the corresponding arc.

Also, because things are so minutely different over the course of a half a degree, Ptolemy carries out the sexagesimal expression one place further, out to the "thirds" column, so that the last entry for each "sixtieth" increment is the number of 216000th parts of a 120th-Part of the diameter (216000 = 3600 × 60).

Here is the example worked out:

Chord 22½ °	= 23 24 39	=	23.4108333
Chord 23°	= 23 55 27	=	<u>23.9241666</u>
Difference of these chords		=	0.5133333
Difference ÷ 30		=	0.01711111
		=	1711/100000
		=	3695.76/216000

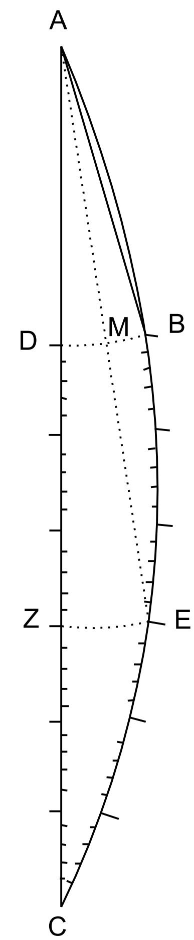
= 0 + 3600/216000 + 60/216000 + 35.76/216000
= 1’ 1’’ 36’’’

which is Ptolemy’s “sixtieth” value for chords after 22½°.

Since we are looking for the chord of 22⅔°, we need to translate the fraction ⅔ into a fraction over 30 (since there are 30 increments over the half-degree span). So we are looking for the chord of 22 20/30°. To do so, we must add to the value of the chord of 22½° *twenty times* the incremental “sixtieth” value we just found. Hence:

Chord of 22⅔° = 23 24’ 39’’ + 20(1’ 1’’ 36’’’)
= 23 24’ 39’’ + 20’ 20’’ 720’’’
= 23 24’ 39’’ + 20’ 20’’ 12(60’’’)
= 23 24’ 39’’ + 20’ 32’’ 0’’’
= 23 44’ 71’’
= 23 45’ 11’’

ILLUSTRATION:

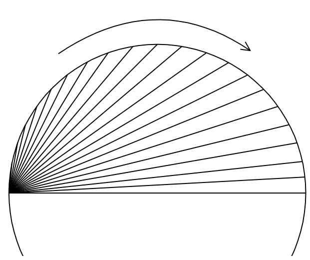


If arc AB = 15°
and arc AC = 15½°
then arc BC = ½° = 30’ (here grossly exaggerated in the picture)

If we make a circle around A with radius AB and cut off AD from AC,
then DC is the difference between the two successive chords.
Chop it into 30 equal parts, and each of those is an increment-of-chord-increase for “sixtieths” of a degree between 15° and 15½°.

Chop up arc BC into 30 equal parts, and each of those is an arc-minute.
If arc BE is 15 of those, then AE is the chord of 15¼°.

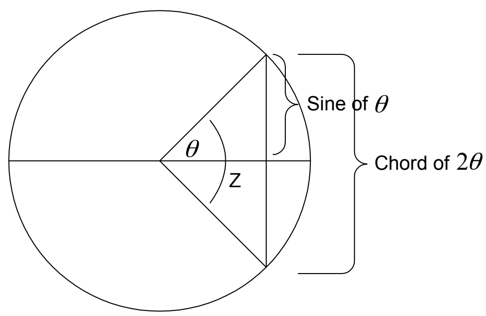
We approximate AE by adding 15 portions of DC to AB, assuming an approximate equality between ME and DZ.



NOTE: Sixtieths *shrink* as we go through the table. Why? Because the difference between two successive chords shrinks as we go through the arcs from 0° to 180°, i.e. they grow more rapidly at the beginning, and hardly differ at all toward the end.

NOTE: Since Ptolemy enables us to find a chord for every arc-minute around a semicircle, that's 60 × 180 chords, i.e. 10,800 chords!

(17) THE RELATION OF PTOLEMY'S TABLE OF CHORDS TO A TABLE OF SINES



The "sine" of an angle θ is found by placing it at the center of a circle whose radius is called "1", and dropping from the end of one radial leg a perpendicular to the other radial leg. The length of that perpendicular is obviously decided by the size of the angle, and it is called the "sine" of that angle, or " $\sin\theta$ ".

Since lines from the center of a circle drawn perpendicular to any line inside the circle will bisect it, plainly if we just double the sine of θ, we get the chord of 2θ,

i.e. $2\sin\theta$ = Chord of 2θ

But there is a difference of units to consider. When expressing the length of the "sine", by convention the radius = 1, but by Ptolemy's convention, radius = 60.

So $60 \cdot 2\sin\theta$ = Chord of 2θ

i.e. $\sin\theta$ = (Chord of 2θ) ÷ 120

and Chord Z = 120 sin (½ Z)

EXAMPLE:

Chord 1° = 120 sin ½° (if our formula is right)
Chord 1° = 1.04718426 (found by calculator)
Chord 1° = 1^P 2' 49.86''

So Ptolemy's table is accurate to within one arc second!
Still, the truth is that

Chord 1° < 1^P 2' 50''

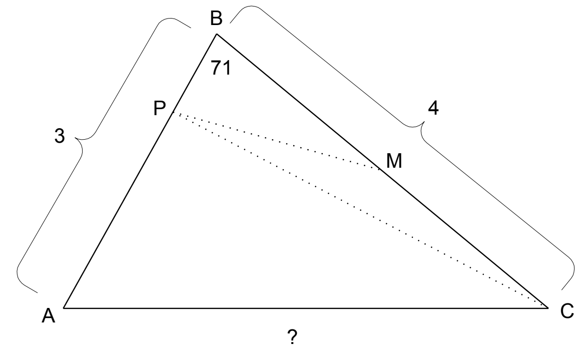
but not by much. This resolution goes way beyond what the eye can distinguish in the heavens; even 8-inch telescopes are limited to about 4 arc seconds of resolution on a good night.

(18) SOLVING TRIANGLES.

The whole point of the Table was to enable us to solve triangles, given sufficient data about them—i.e. given certain sides or angles, to be able to determine all the remaining sides and angles.

We should now solve some triangles in order to appreciate the power the Table has now given us.

PROBLEM 1: "2 SIDES & INCLUDED ANGLE"



GIVEN: $\triangle ABC$
$AB = 3$
$BC = 4$
$\angle ABC = 71°$

FIND: AC

Drop CP perpendicular to AB.
Hence the circle on diameter BC, center M, passes through P.
And all angles in $\triangle BPC$ are given (i.e. $\angle BCP = 19°$).
So PC is the chord of $\angle PMC$.
Now $\angle PMC$ is double 71°, i.e. double $\angle PBC$ (i.e. $\angle ABC$), since they stand on the same chord but one is at the center and the other at the circumference.
So $\angle PMC = 142°$.
So PC is the chord of 142°.
So PC is given on our table of chords, but expressed in units where BC = 120.

So when	BC = 120
then	PC = 113 27' 44'' = 113.46222

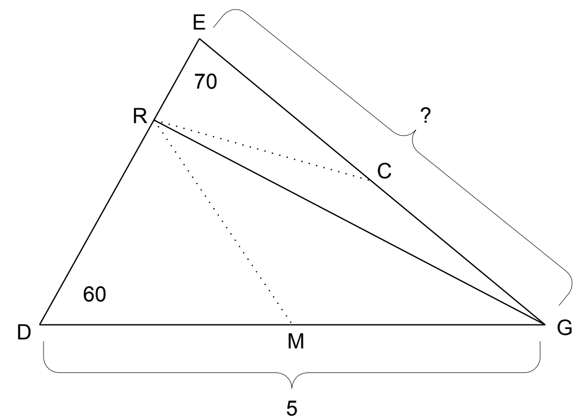
Hence when	BC = 4
then	PC = 3.782074 (just maintaining the same ratio)

Since we now know BC and PC, we can solve for PB by the Pythagorean Theorem. And since we already know AB, we can find also the difference, AP. And since we know PC, we can now find AC by the Pythagorean Theorem.

So AC has been found.

(As an exercise, finish the actual calculations, and find the value of AC.)

PROBLEM 2: "2 ANGLES & ANY SIDE"



GIVEN: △DEG
∠DEG = 70°
∠EDG = 60°
DG = 5

FIND: EG

Drop GR at right angles to DE.
Hence the circle on diameter DG, center M, will pass through R.
So RG is the chord of ∠RMG.
But ∠RMG is double ∠RDG, i.e. ∠RMG = 120°.
So RG is the chord of 120°, and we can find the value of RG on Ptolemy's Table, expressed in such units that DG is 120.

So when	DG = 120
then	RG = 103 55' 23'' = 103.923

Hence when	DG = 5
then	RG = 4.330125

Again, the circle on diameter EG, center C, passes through R.
So RG is the chord of ∠RCG.
But ∠RCG is double ∠REG, i.e. ∠RCG = 140°.
So RG is expressed as the chord of 140° where EG is 120.

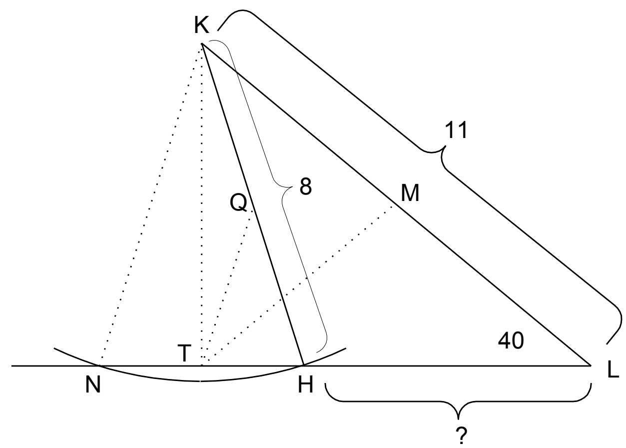
So when	EG = 120
then	RG = 112 45' 48'' = 112.76333

So when	RG = 4.330125 (which is when DG = 5)
then	EG = 4.608014

PROBLEM 3: "2 SIDES & NON-INCLUDED ANGLE"

GIVEN: △KHL
KL = 11
KH = 8
∠KLH = 40°
∠KHL is obtuse

FIND: HL



It might seem that "2 sides and a non-included angle" is insufficient information to determine a triangle—but it is almost sufficient. That is, only 2 triangles can have those specifications. Consider △HKL, in which ∠KHL is obtuse. If we extend LH and draw a circle around K as center with radius KH (= 8), that circle will obviously cut LH produced, and will cut it only once, say at N.

So △KHL and △KNL are the only 2 triangles which can have the angle at L equal to 40°, side LK = 11, and the other side from K = 8.

And since ∠KNL must be acute, we have fully specified which triangle we are talking about once we say that ∠KHL is obtuse.

To find HL, drop KT at right angles to NH (hence NT = TH).
Now the circle on diameter KL, center M, passes through T.
Therefore KT is the chord of ∠KMT, i.e. of double ∠KLH, i.e. KT is the chord of 80°. Hence the value of KT is known (from Ptolemy's table) when KL = 120. Adjusting proportionally, we will know the value of KT when KL = 11 (which is also when KH = 8).

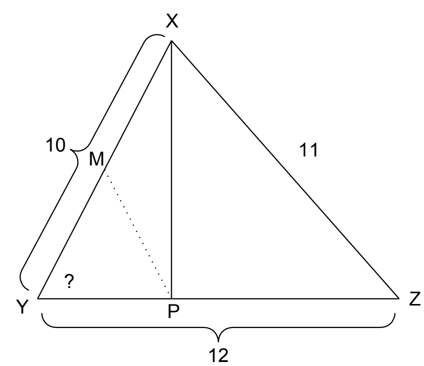
Since we know KH and KT, using the Pythagorean Theorem we can also find HT.

Again, the circle on diameter KH, center Q, passes through T.
Therefore HT is the chord of ∠HQT.
But we know the value of HT when KH is 8 (we just found it above).
Using a proportion, we can find the value of HT when KH is 120.
At that value, HT will occur on Ptolemy's Table as the chord of ∠HQT, and therefore we now know ∠HQT.
But ∠HKT is half of that, so we know that angle, too.
And ∠TKL = 50°.
So ∠HKL, their difference, is now known.

We now know ∠HKL, and we also know the sides about it, KH and KL (which are 8 and 11). Hence we can use the technique of Problem 1 above to find HL.

EXERCISE: Follow through and calculate the actual numerical value of HL.

PROBLEM 4: "3 SIDES"



GIVEN: △XYZ
XY = 10
XZ = 11
YZ = 12

FIND: ∠XYZ

Drop XP at right angles to YZ.

Now $XZ^2 = YZ^2 + XY^2 - 2\,(ZY.YP)$ [Euc. 2.13]

i.e. $11^2 = 12^2 + 10^2 - 2 \cdot 12 \cdot YP$

so $121 = 144 + 100 - 24 \cdot YP$

so $24 \cdot YP = 244 - 121$

so $YP = 5.125$

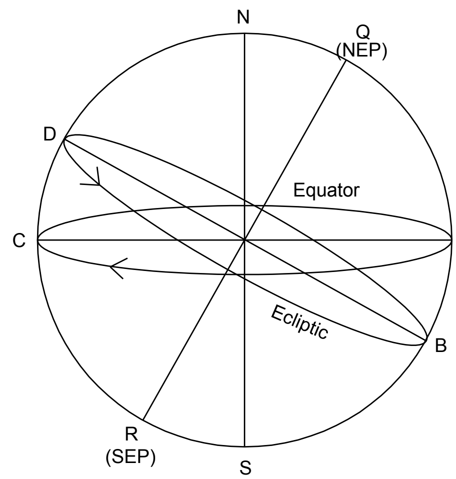
But the circle on diameter XY, center M, passes through P.
Therefore YP is the chord of ∠PMY where XY = 120.
But where XY = 120, YP = 61.5.
Looking up 61.5 on Ptolemy's Table will give us ∠YMP.
Half this angle will give us ∠YXP.
Subtracting ∠YXP from 90° will give us ∠XYZ, which is sought.

EXERCISE: Follow through and calculate the actual numerical value of ∠XYZ.

PTOLEMY

DAY 9

ON THE ARC BETWEEN THE TROPICS



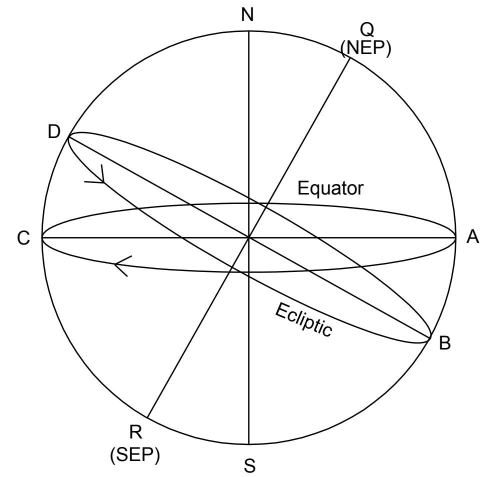
In Chapter 12 of Book 1 of his *Almagest*, Ptolemy gives us the first numerically quantified detail in his theory. We have seen that there are two important great circles on the celestial sphere, namely the celestial Equator and the Ecliptic, corresponding to the first and second primary motions in the heavens. What we are looking for is the inclination between these, that is, the greatest circular arc intercepted between them. Suppose we draw the 4-Pole circle through the North and South celestial poles (N and S) and through the North Ecliptic Pole and the South Ecliptic Pole (Q and R). An arc of this great circle will be cut off between the equator and the ecliptic. Call this arc AB (or CD on the other side). How long is this arc? How many degrees is it? That is the first numerical detail we will establish in the Ptolemaic geocentric model.

Ptolemy is actually looking for "the arc between the tropics." A "**tropic**" is a "turning point," a place where the Sun turns back from its climb North or from its drop South, and stars going the opposite way. In the accompanying figure, we must picture the whole sphere spinning about the celestial axis, NS, in the direction indicated by the arrow drawn on the Equator. It does that once every 24 hours or so. At the same time, however, we must picture the Sun on the Ecliptic (actually, *projected onto it*, since its true orbit lies well inside the sphere and much closer to the Earth than to the sphere of fixed stars), being carried around with this motion, but at the same time *crawling in the opposite direction along the Ecliptic*, in the direction indicated by the arrow on the Ecliptic. When the Sun is up at D, the northernmost point in its trip around the Ecliptic, it will start to come South afterward, and so D is a "turning point," a "tropic." So D is the "**summer tropic**," because for those of us living in the northern hemisphere, that will be the longest day of the year. And B, the southernmost point in the Sun's annual journey, is the "**winter tropic**," the shortest day of the year for those of us in the northern hemisphere, but the longest for those in the southern hemisphere. Ptolemy is looking for the sum of the two arcs, AB + CD, since that is the total amount of arc separating the two tropic points. That sum, in other words, represents the total amount of North-South deviation in the Sun's annual motion.

Ptolemy also takes "the arc between the tropics" to be equal to the length of the arc between the poles of the Equator and of the Ecliptic.

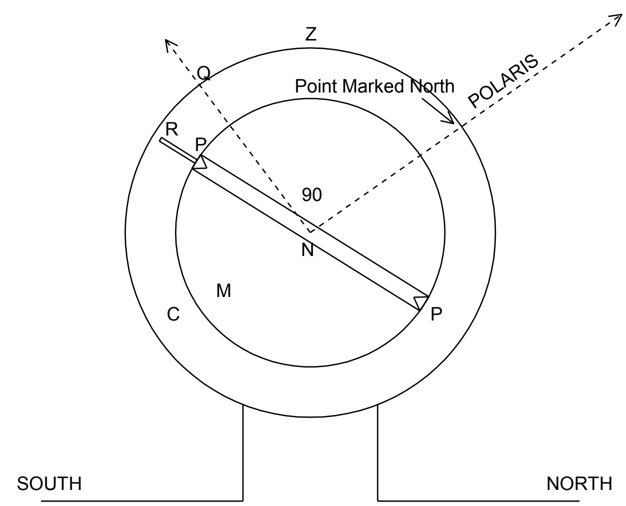
That is,

arc AB = arc CD = arc NQ = arc RS



That is obvious just from the fact that NS and QR are each at right angles to AC and BD respectively. Anyway, what Ptolemy actually observes is: arc AB + arc CD, i.e. the amount of arc between the sun's northernmost and southernmost points along the Ecliptic.

His instrument for doing this is simply a pair of concentric and carefully made circles (e.g. made of brass) with the circular faces set up in the plane of his meridian. That is, the faces of the circles (graduated, with degrees, minutes, and seconds marked out on them) are plumb to the horizon, and parallel to the North-South line.

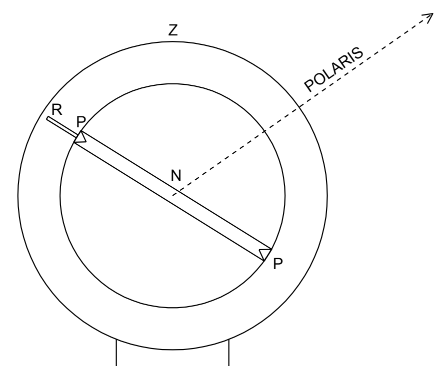


PTOLEMY'S MERIDIAN DEVICE

C = fixed brass graduated circle in plane of the meridian
M = circle rotating in plane of meridian; spins about center
N = center of circles
P = prism, standing proud for casting a shadow
R = pointer fixed to the upper prism

The graduated circle has celestial north marked out, i.e. the line from N to the point marked "North" on the circle points straight at the star Polaris. The equator is also marked out—NQ points to where it cuts the meridian, i.e. NQ points right to the celestial equator, and hence the line QN is at right angles to the line from N to Polaris.

With this simple tool, he can measure the celestial latitudes of heavenly bodies.



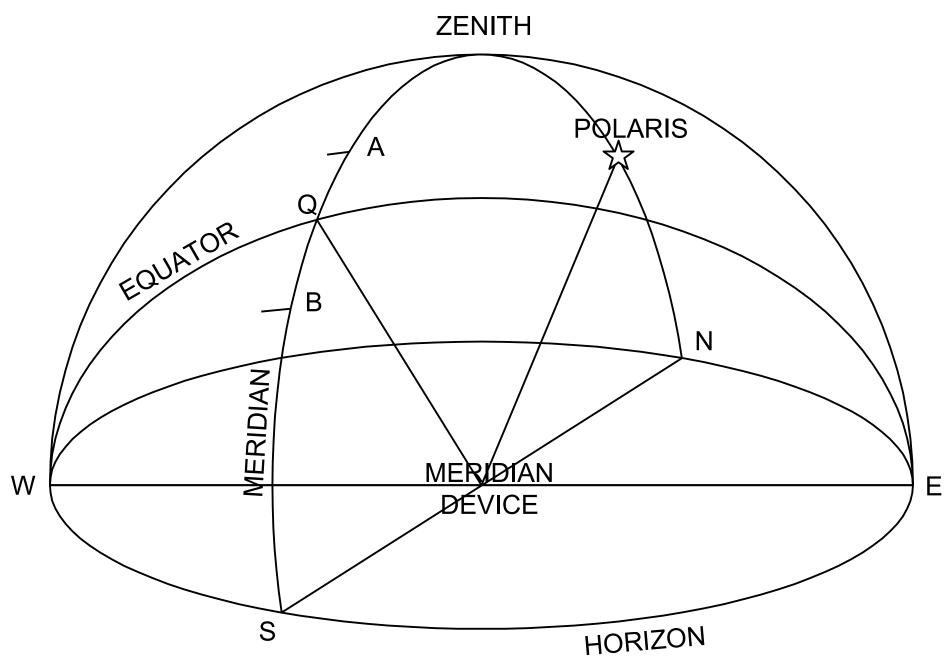
When the sun is at his meridian, he moves the inner circle until the shadow of the upper prism hits the lower prism, so that PNPR points right at the sun. The device then registers the sun's latitude. (It is handy to use shadows to point the pointer right at the Sun, since it is dangerous to look right at the Sun.)

If we do this every day (or nearly every day) for a year (or even for several years), we will be able to mark out exactly where the sun cuts our meridian on the summer and winter solstices (we already know that on the equinoxes it is on the celestial equator, and hence cuts our meridian at Q).

A = where summer tropic cuts meridian
Q = where sun cuts meridian at equinox
B = where winter tropic cuts meridian
Z = zenith

Ptolemy finds that arc AB = 47° 40' (about)

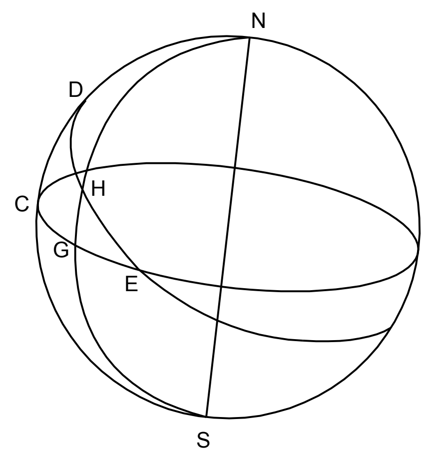
Half of that is about 23½° , which is the inclination of the ecliptic to the celestial equator.



PTOLEMY

DAY 10

PREPARING TO FIND OTHER ARCS BETWEEN THE ECLIPTIC AND EQUATOR



You might have noticed that in Day 9, when we found the arc between the tropics, we made no use of our table of chords. We simply observed the arc, watching the Sun's latitude throughout the year. But Ptolemy will next show us how to find other arcs intercepted between the ecliptic and equator. For example, if E is the place where the ecliptic and equator intersect, and arc EC and arc ED are each 90°, then the arc of the great circle through C and D will be half the arc between the tropics, or about 23½° . But what if arc EG on the equator is something less than 90°, and we draw a great circle through the celestial poles and through G, intercepting the ecliptic at H? How great will arc GH be? Will we be able to tell, if we are given arc EG in degrees?

We will. And one can see how this kind of power of getting the values of arcs on a great circle given other arc-values on the same sphere could be very useful in astronomy. And to do this, we will need to use our table of chords and arcs. But first we will need to develop a little more geometry! We will need some Lemmas about spherical geometry, and in order to prove those, we will need first a few Lemmas concerning plane geometry. And since we will need to "**compound ratios**" in order to understand these Lemmas, and since that mathematical operation is extremely common in Ptolemy (and in Copernicus and in Kepler, and also in Apollonius, whose geometry Kepler presupposes to some extent), we will have to begin with a refresher on what it means to "compound ratios." So here is our agenda for today:

(1) Review what it means to compound ratios.
(2) Prove the "Menelaos Theorems" as Lemmas preliminary to the spherical proofs.
(3) Prove the "Spherical Menelaos Theorems" as Lemmas preliminary to determining the values of certain arcs on the celestial sphere.
(4) Apply these Menelaos Theorems to an astronomical problem, namely finding the length of the longest day of the year where you live.

We will cover Step (1) today, Step (2) on Day 11, Step (3) on Day 12, and Step (4) on Day 13.

(1) COMPOUNDING RATIOS.

Suppose you have two ratios, ab : bc & ef : fd. There is any number of ways one could unite these terms so as to produce a new ratio. For example:

ab + ef : bc + fd

That is a new ratio, formed out of the terms of the original ratios. And if one liked, one could call this new ratio "the ratio compounded of the original ratios." But there is a reason we don't do this. If you start with two ratios and go through this process, and if I start with two ratios that are the same as yours (but perhaps expressed in different numbers), and then I go through the same process, then your result and my result might not be the same! The sameness of my starting ratios with yours, and the sameness of the process, does not guarantee the sameness of the result, which makes the process not very useful. For example, let the starting ratios be

1 : 2 & 3 : 4

Then the new ratio, formed by adding the antecedents and adding the consequents, is

4 : 6

But now suppose we start with the same ratios as before, but this time expressed in different numbers, such as

2 : 4 & 9 : 12

Then the new ratio, formed the same way, will be

11 : 16

which is certainly not the same as 4 : 6.

But there is another way of producing a new ratio from two given ratios which always gives the same result, regardless of the terms in which the original ratios are expressed. Moreover, this way of combining ratios (which I am about to describe) is important, because in mathematics and in nature there are many interesting truths which involve this way of producing new ratios. Hence this way of producing ratios deserves to be named, and so we name it "compounding ratios."

The definition of "compounding ratios" is this: Given any two ratios, a : b & c : d, find another pair of ratios the same as these *but in which the consequent of the first ratio is the same as the antecedent in the second*, and then form the ratio of the extremes. The new ratio is said to be the "compound" of the original two ratios.

For example, given

1 : 2 & 3 : 4

we can take two ratios the same as these in which the consequent of the first ratio will be the same as the antecedent in the second, like this:

3 : 6 & 6 : 8

and the ratio 3 : 8 is said to be the "compound" of the original two ratios. One notation for this is as follows:

3 : 8 :: (1 : 2) comp (3 : 4)

or 3 : 8 = (1 : 2) c (3 : 4)

Notice that the compound ratio is the same regardless of the numbers in which the original two ratios are expressed. We could express them thus:

2 : 4 & 9 : 12

and if we go through the process again, compounding these ratios, we can still use

3 : 6 & 6 : 8

and thus get 3 : 8 as the compound ratio. Or, if we like, we can use

18 : 36 & 36 : 48

so that the compound result is 18 : 48, which, simplifying, is the same ratio as 3 : 8.

If we are dealing with integers, then there is an easy, always-works method for forming the compound ratio. All we have to do is take the ratio of the product of the original antecedents to the product of the original consequents. For example, given

4 : 5 & 2 : 3

the compound ratio must be (4 × 2) : (5 × 3). The reason for this is that we can get the consequent of the first ratio to be the same as the antecedent of the second (which is the key step to compounding) by multiplying the terms of the first ratio by 2 and the terms of the second ratio by 5, giving us

8 : 10 & 10 : 15

But this means the new extremes will be 4 × 2 and 5 × 3, and hence their ratio will be the "compound" of the original ratios, by the definition.

From this, one can begin to see why people say that "compounding ratios is the equivalent of multiplying fractions." If the fractions are ratios of integers, then the new fraction resulting from multiplying the original two will be the same as the compound ratio. For example:

$$\frac{4}{5} \times \frac{2}{3} = \frac{8}{15}$$

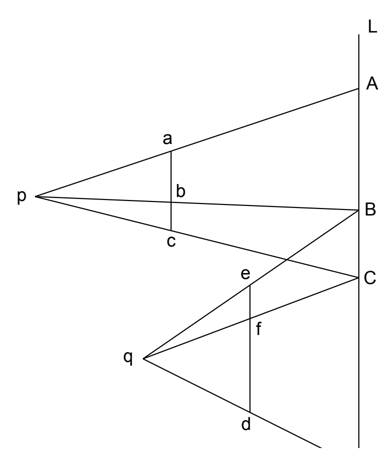
and (4 : 5) c (2 : 3) = 8 : 15

I say "begin to see" rather than just "see," however, because it is not clear how to "multiply" things such as √2 and √3 (our given ratios might consist of such irrational terms), although there is no difficulty compounding ratios with terms such as these. To multiply √2 by √3, one would first have to learn a new definition of "multiplication" besides "taking 2 three times and adding them all up" (as one does for 2 × 3), since there is no clear sense to "taking √2 the square-root-of-three-times." (As it turns out, the way to define the multiplication of irrationals will be by completing a proportion among straight lines. So "compounding ratios" is more basic and intelligible in itself than "multiplying fractions," since one can define "compounding ratios" without thinking of "multiplying fractions," but not the reverse, except when the fractions happen to have integral terms. But this is a digression.)

Note also that it is possible to compound ratios which are not of comparable things. If the original ratios to be compounded are a : b & c : d, and a : b is a ratio of volumes, while c : d is a ratio of areas, then we can still compound these so long as we can find two other ratios, the same as these, in which the terms *are* comparable. For instance, suppose we can find a pair of straight lines A : B which have the same ratio as a : b, and we can also find a pair of straight lines C : D which have the same ratio as c : d. Then we can compound A : B and C : D the normal way, and the resulting ratio is the compound of the original ratios a : b and c : d as well.

There is more than one way to present an image of compounding ratios to make it more memorable.

One way is by the accompanying diagram.



If our two original ratios are ab : bc & ef : fd, let abc and efd be drawn parallel to one another, and draw any third line L parallel to them also. We will use this line L to form the ratio compounded of the two originals.

Pick some point P not in line with abc and join pa, pb, pc and extend these to points A, B, C on L. Clearly AB : BC = ab : bc.

Now join Be, Cf, and extend these until they meet at a point, q. Join qd and extend it till it cuts L at a point D. Clearly BC : CD = ef : fd.

But by the definition of compounding,

AB : CD = (AB : BC) c (BC : CD)

And therefore, by the sameness of these ratios with the originals,

AB : CD = (ab : bc) c (ef : fd)

Another image which helps to understand the compounding of ratios is as follows. Suppose you have two maps, Map 1 and Map 2, and each one depicts roads in the correct ratios (i.e. the same ratios that the roads themselves have in reality), but neither one tells you how much real distance an inch on the map represents. There is no indication of scale, in other words. Now you see that

on Map 1 AB : BC = 3 : 2

which you determine with a ruler, measuring the relative lengths of the roads as depicted on that map. Again

on Map 2 BC : CD = 12 : 4

Can you conclude, then, that

AB : CD = 3 : 4 ?

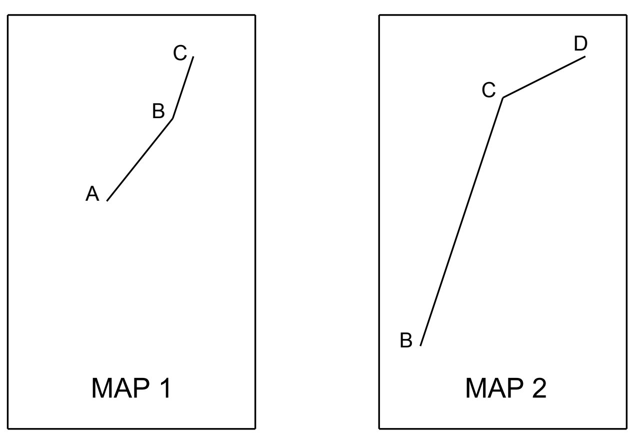
By no means. The scales of the maps are clearly different, since one and the same road, road BC, is represented as 2 inches on Map 1, but as 12 inches on Map 2. To adjust for this difference of scale, in order to find the ratio of AB : CD from the two given ratios, you must compound the ratios:

AB : CD = (AB : BC) c (BC : CD) [by def. of compounding]

so AB : CD = (3 : 2) c (12 : 4)

so AB : CD = 36 : 8

so AB : CD = 9 : 2



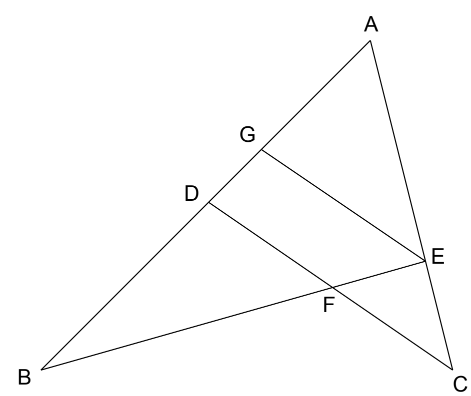
PTOLEMY

DAY 11

PREPARATORY LEMMAS FOR SPHERICAL MENELAOS PROOFS

On our way to learning how to determine arcs of great circles on a sphere (given the lengths of certain other arcs, or chords, in degrees or in 120^{th} parts of the diameter of the sphere), we need several Lemmas. The first two are called the "Menelaos Theorems," in which we will be compounding ratios.

FIRST LEMMA: MENELAOS 1.



Given: 2 straight lines AB and AC,
2 more drawn to them, BE and CD, cutting each other at F.

Prove: AC : AE = (CD : DF) c (BF : BE)

Draw: EG parallel to CD.

Well,	AC : AE = CD : EG	[Euc. 6.4]
but	CD : EG = (CD : DF) c (DF : EG)	[def. of compound ratio]
so	AC : AE = (CD : DF) c (DF : EG)	
i.e.	AC : AE = (CD : DF) c (BF : BE)	[Euc. 6.4]

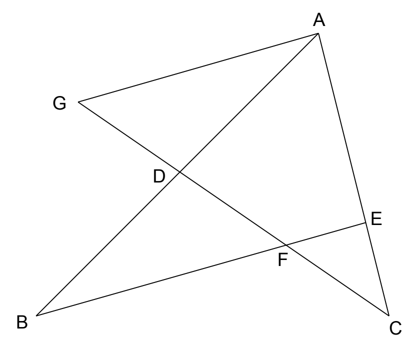
Q.E.D.

NOTE: This Lemma, uninspiring as it might seem in itself, is used to prove many interesting theorems in geometry, e.g. Desargue's Theorem, Pascal's Mystic Hexagon Theorem, and many more.

The figure (leaving out the auxiliary line EG) is what you might call a "Star Trek figure." So I call it, anyway. It can also be called a "Menelaos figure," but that's not as fun.

This figure includes 4 triangles—2 overlapping big ones, and 2 little ones at the bottom. If you circumscribe a circle around each of these 4 triangles, the 4 circles all pass through the same point, P, called the "Miquel Point."

These are all asides, but sometimes it is good to stop and smell the roses.



SECOND LEMMA: MENELAOS 2.

Given: Same diagram, but with AG parallel to EF

Prove: CE : AE = (CF : DF) c (BD : AB)

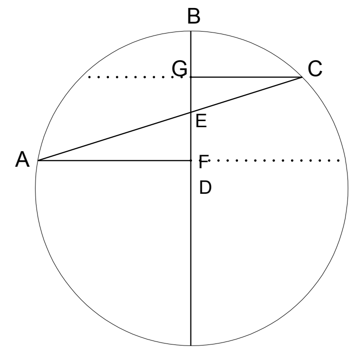
Well,	CF : FG = CE : AE	[Euc. 6.2]
but	CF : FG = (CF : DF) c (DF : FG)	[def. of compound ratio]
so	CE : AE = (CF : DF) c (DF : FG)	
i.e.	CE : AE = (CF : DF) c (BD : AB)	[see below]

Q.E.D.

How do we know that DF : FG = BD : AB ?

Well,	DF : BD = GD : DA	[△BDF similar to △GDA]
so	DF : BD = GD + DF : DA + BD	[Euc. 5.18, *componendo*]
so	DF : BD = GF : AB	
so	DF : GF = BD : AB	[alternately]

THIRD LEMMA:



Given: Circle, center D, points A, B, C on the circumference,
arc AB < 180°
arc BC < 180°
AC joined, DEB joined

Prove: AE : CE = chord of 2(arc AB) : chord of 2(arc BC)

Draw: AF perpendicular to DB
CG perpendicular to DB

Well, AF : CG = AE : CE [Euc. 6.4]

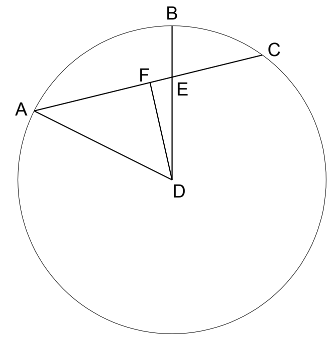
but AF : CG = chord of 2(arc AB) : chord of 2(arc BC)

so AE : CE = chord of 2(arc AB) : chord of 2(arc BC)

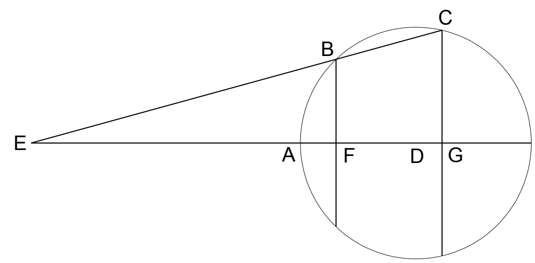
Q.E.D.

Here is a quick preview of the NEXT THREE LEMMAS, before we actually prove them:

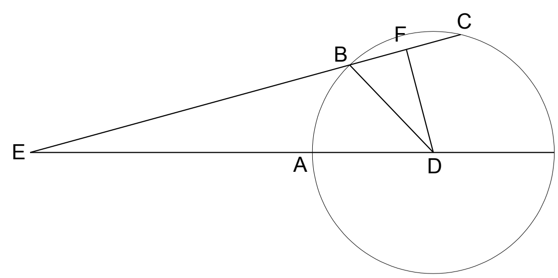
[1] If arc AC is given in degrees, and the ratio of [the chord of 2 arc AB] to [the chord of 2 arc BC] is given in numbers, then arcs AB and BC are given.



[2] If arcs AB and AC are each less than 180°, CB joined and extended till it meets DA produced at E, then CE : BE = chord [2 arc AC] : chord [2 arc AB].



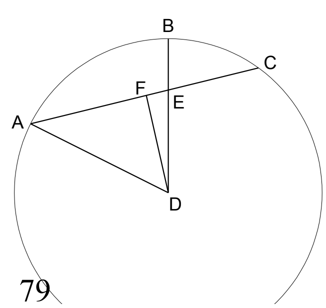
[3] If arc BC is given, and the ratio of [the chord of 2 arc AC] to [the chord of 2 arc AB] is given in numbers, then arc AB is given.



NOTE: The proofs for these work just as well if the arcs are not less than 180°, I believe, but Ptolemy specifies that the arcs are less than 180°.

ALSO: What does Ptolemy mean here by "given"? When he says that we are "given" a chord length, he means we are given (or else able to compute) its numerical value in units of the 120th parts of the diameter of the circle it is in. When he says we are "given" an arc length, he means we are given (or else able to compute) its numerical value in degrees, minutes, and seconds.

FOURTH LEMMA:



Given:	arc AC in degrees ratio of [chord of 2 arc AB] : [chord of 2 arc BC] in numbers		
Prove:	arcs AB and BC are given in degrees		

Well, ∠ADC is given [subtends arc AC]
so ½ ∠ADC is given
so ∠ADF is given [if we draw DF perpendicular to AC]

but, by our previous theorem,

AE : CE = [chord of 2 arc AB] : [chord of 2 arc BC]

and the ratio on the right is given, hence it is also given *componendo*:

AE + CE : CE = [chd of 2 arc AB] + [chd of 2 arc BC] : [chd of 2 arc BC]

Now AE + CE is just AC, and AC is given by the Table, because the arc AC is given. But the ratio on the right is given. Hence CE, the remaining term, is also given.

So	AC – CE	given	
i.e.	AE	given	
but	AF	given	(just half of AC)
so	AE – AF	given	
so	EF	given	
but	DF	given	(AF given, AD radius, right △AFD)
hence	DE	given	(EF & DF given in right △DEF)

So DF, FE, ED are all given in terms of the 120^{th} parts of the whole circle of radius AD. So we know their ratios. Hence, now calling DE "120," we can translate DF & FE into its terms, and using our table of chords & arcs, we know ∠EDF.

So	∠EDF + ∠FDA is now given		
so	∠EDA	given	
i.e.	∠ADB	given	
i.e.	arc AB	given	
so too	arc BC	given	(since arc BC = arc AC – arc AB)

Q.E.D.

FIFTH LEMMA:

Given: arcs AB and AC each less than 180°
join CB, DA, produce to E

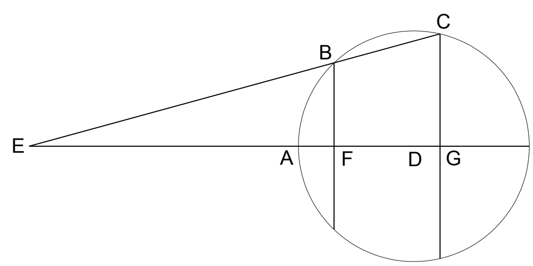
Prove: CE : BE = chord [2 arc AC] : chord [2 arc AB]

Well, CG : BF = CE : BE

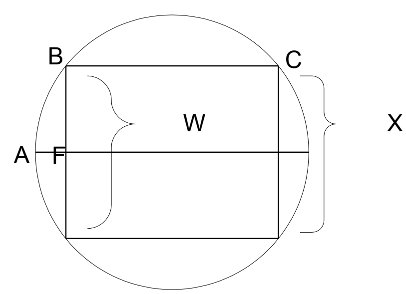
but CG : BF = chord [2 arc AC] : chord [2 arc AB]

so CE : BE = chord [2 arc AC] : chord [2 arc AB]

Q.E.D.



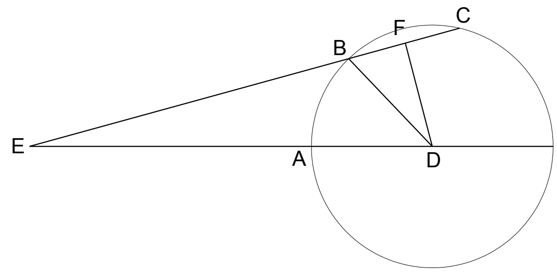
QUESTION: What if BC is parallel to DA, so there is no point E at which they meet? In that case, chord [2 arc AC] = chord [2 arc AB], i.e. the chords W and X are equal.



SIXTH LEMMA:

Given: arc BC in degrees
numerical ratio of chord [2 arc AC] : chord [2 arc AB]

Prove: arc AB given in degrees



Well,	arc BC	given
so	∠BDC	given
so	½ ∠BDC	given
so	∠BDF	given

Also	arc BC	given
so	BC	given by Table
so	BF	given (it's just ½ BC)
and	BD	given (60)
so	DF	given (BD & BF given, △BDF is right)

But CE : BE = chord [2 arc AC] : chord [2 arc AB] (previous)

But that ratio of chords is given numerically, so it is also given *separando*

CE – BE : BE = chd [2 arc AC] – chd [2 arc AB] : chd [2 arc AB]

i.e. BC : BE = chd [2 arc AC] – chd [2 arc AB] : chd [2 arc AB]

so BE is given (since BC is given, and so is the ratio on the right)

So BE + BF = EF given
so ED given (EF & FD given, △DEF is right)

So, calling ED "120," we can translate EF into 120th parts of ED, and therefore ∠EDF is known by our table of chords & arcs.

But	∠BDF	given
so	∠EDF – ∠BDF	given
i.e.	∠ADB	given
hence	arc AB	given

Q.E.D.

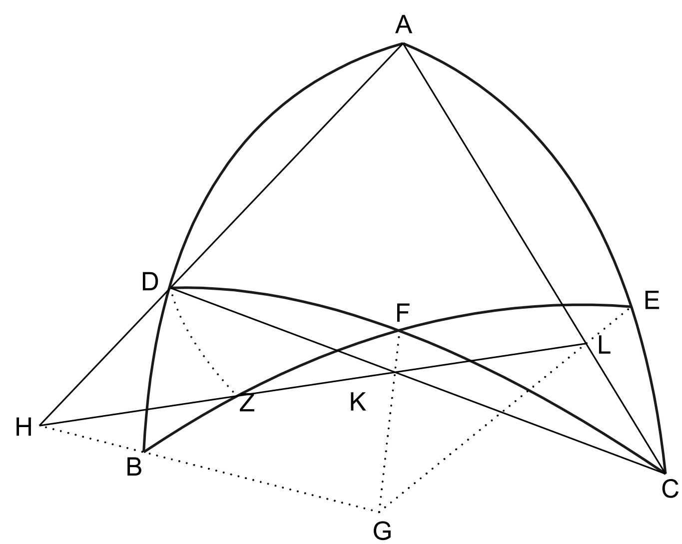
PTOLEMY

DAY 12

SPHERICAL MENELAOS PROOFS

Next we are ready to prove the Spherical Menalaos Theorems, which are analogous to the Menelaos Theorems we proved about plane triangles, except now we are going to be dealing with "spherical triangles," or triangle-like figures on the surfaces of spheres. This is on our way to being able to determine the values of chords in a sphere given certain arcs of great circles in degrees, or, conversely, being able to determine the values of circular arcs on the surface of a sphere given the lengths of certain chords in 120^{th} parts of the diameter of the sphere. The following material is developed by Ptolemy in Chapter 13 of Book 1 of his *Almagest*.

SPHERICAL MENELAOS PROOF



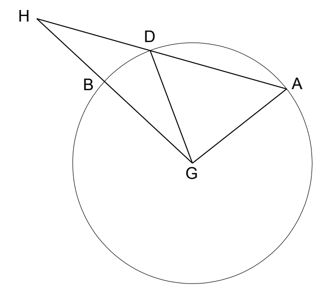
In the figure we are looking at arcs on the surface of a SPHERE, so we are looking at a kind of SPHERICAL STAR TREK FIGURE, or spherical Menelaos Figure.

Let arcs AB, AC, CD, BE all be portions of great circles on a sphere with center G (imagine G as down below the page, further away from you than the spherical surface bulging out at you). What we want to prove is that

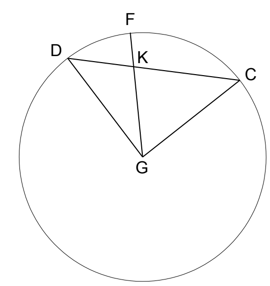
Chd[2arcCE] : Chd[2arcAE] = Chd[2arcCF] : Chd[2arcDF] COMP Chd[2arcBD] : Chd[2arcAB]

Now, since G is the center of the sphere, we can consider it as the center of the different arcs of the great circles, and looking at these one at a time:

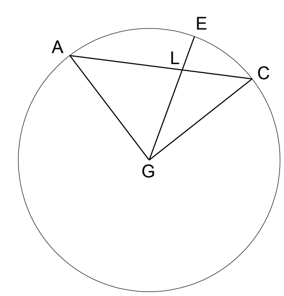
Join GB, GD, GA, AD.
Produce AD & GB (so they must meet at some point H).



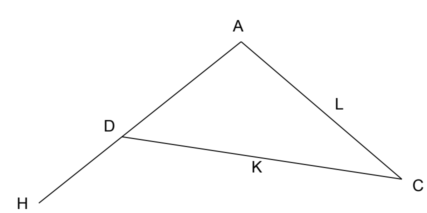
Join GC, GF, CD.
So CD & GF must cut, say at K (so K is on DC and on GF).



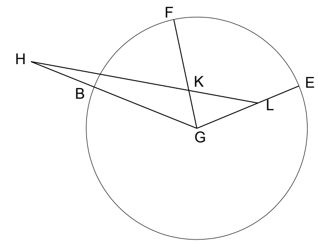
Join GE, AC.
So these must cut, say at L (so L is on GE and on AC).



Now H, K, L are all in the plane of △ACD:
H is on AD produced,
K is on DC
L is on AC

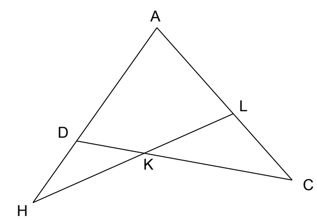


And H, K, L are all in the plane of circle BFE:
H is on GB produced,
K is on GF
L is on GE



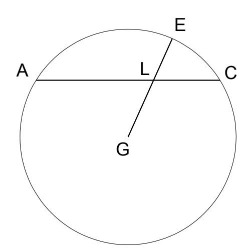
Hence H, K, L are all in 2 different planes, and hence they all lie on the one straight line which is the common section of those two planes.

Hence we have a Menelaus figure:

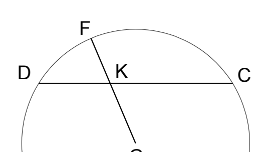


So CL : AL = (CK : DK) c (DH : AH) [Lemma 2, Day 11]

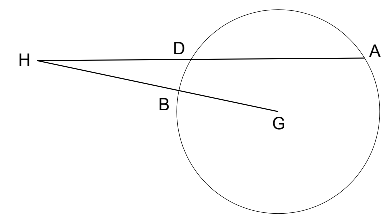
But, CL : AL = chd[2arcCE] : chd[2arcAE] [Lemma 4, Day 11]



and CK : DK = chd[2arcCF] : chd[2arcDF] [Lemma 4, Day 11]



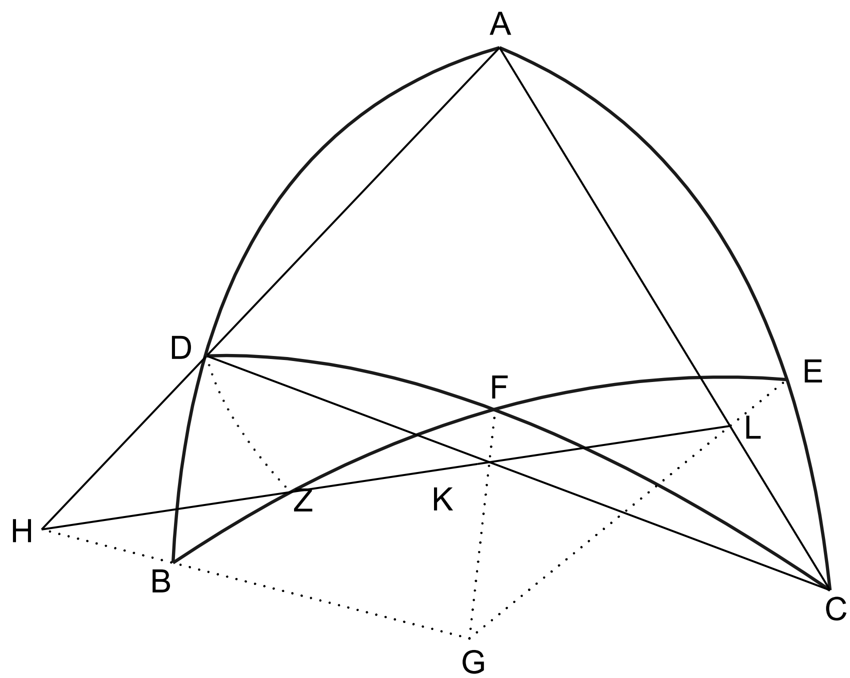
and DH : AH = chd[2arcBD] : chd[2arcAB] [Lemma 5, Day 11]



So, substituting all those ratios of chords into our proportion:

Chd[2arcCE] : Chd[2arcAE] = Chd[2arcCF] : Chd[2arcDF] COMP Chd[2arcBD] : Chd[2arcAB]

Q.E.D.



The main difficulties here are:

(1) Understanding the 3-D diagram.

(2) Remembering that we are compounding ratios of straight lines, not arcs, since we are taking ratios of CHORDS of arcs.

(3) Remember that we are not talking about chords of the arcs themselves, but of double those arcs!

As for getting the diagram straight:

BG, FKG, ELG each pierce the sphere to its center.

H is above the spherical surface.

The Menelaus figure ADHKCL is a plane that pops out of the sphere along arc ZD, peaking at H.

LIMITATION TO THE PROOF.

Ptolemy's argument depends on the Menelaos figure ALCKHD. But such a figure does not necessarily arise just given that AEC, ADB, EFB, DFC are great arcs on a sphere. For example, it could happen that AD and KL are parallel to each other, and then we cannot apply the prior Menelaos lemma as he does in his proof. This should not trouble us too much—in all these cases, the theorem still holds good, only the proof is a little different, and usually simpler. If this were a course in pure geometry, we would want to find the simplest argument that covers all cases. But we want this result for the sake of astronomical applications. So onward!

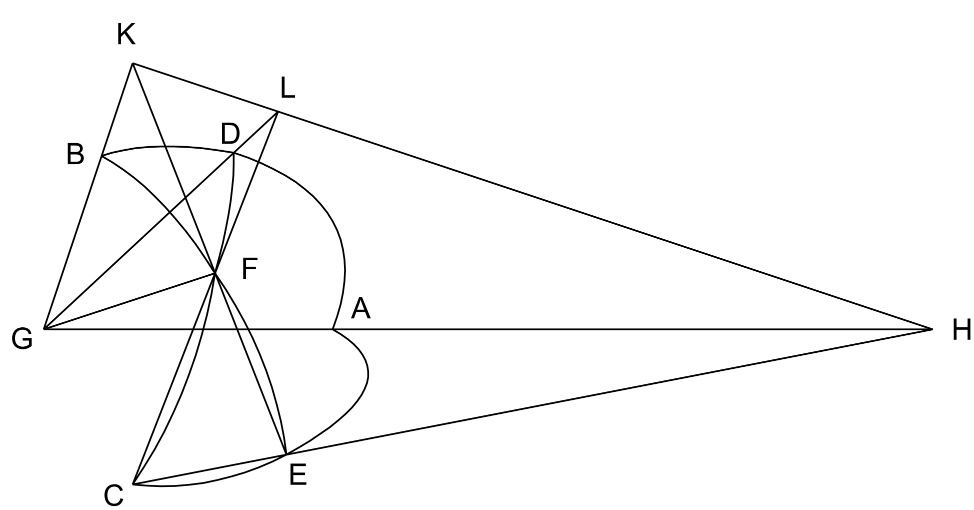
THE OTHER SPHERICAL PROOF.

We have just shown that

Chd[2arcCE] : Chd[2arcAE] = Chd[2arcCF] : Chd[2arcDF] COMP Chd[2arcBD] : Chd[2arcAB]

Ptolemy notes that by similar methods we can also prove that

Chd[2arcAC] : Chd[2arcAE] = Chd[2arcCD] : Chd[2arcDF] COMP Chd[2arcBF] : Chd[2arcBE]



To see this, we start with the same arcs of great circles on a sphere of center G, but now we need a slightly different construction:

Let GA, CE meet at H.
Let CF, GD meet at L.
Let EF, GB meet at K.

Now H lies in both the plane of circle BAG and the plane of △FEC.
And L lies in both the plane of circle BAG and the plane of △FEC.
And K lies in both the plane of circle BAG and the plane of △FEC.

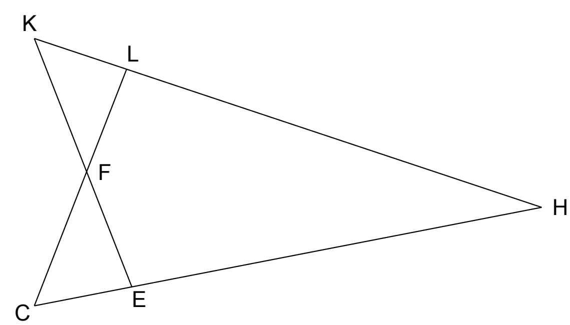
So H, L, K lie in one straight line.
So HLKFCE is a plane Menelaos figure.

Thus CH : EH = (CL : FL) c (FK : EK) [Lemma 1, Day 11]
But CH : EH = Chd[2arcAC] : Chd[2arcAE] [Lemma 5, Day 11]
And CL : FL = Chd[2arcCD] : Chd[2arcDF] [Lemma 5, Day 11]
And FK : EK = Chd[2arcBF] : Chd[2arcBE] [Lemma 5, Day 11]

Thus

Chd[2arcAC] : Chd[2arcAE] = Chd[2arcCD] : Chd[2arcDF] COMP Chd[2arcBF] : Chd[2arcBE]

Q.E.D.



NOTE. This proof suffers from the same limitations as the first proof—but again, that does not matter much, since the theorem is true generally, even when the plane Menelaos figure cannot be constructed.

USEFULNESS. These will obviously be useful for solving for any one of these chords if all of the other chords are known, or at least if the ratios are known. And if the chord is known, then also the arc is known, thanks to our Table of Chords.

BEAUTY. The spherical Menelaos figure is not exactly analogous to the Star Trek figure, since the proportions proved are not about the arcs themselves which constitute the spherical Menelaos figure, *but* there is some surprising analogy, since the proportions among the chords follow the same order as the proportions among the sides of the Star Trek figure.

That is, if we conceive of a plane Star Trek figure with points AECFBD analogous to those in our spherical figure, it is true in the plane figure that

CE : AE = (CF : DF) c (BD : AB)

and AC : AE = (CD : DF) c (BF : BE)

where the terms are all in the same order as in these spherical proofs.

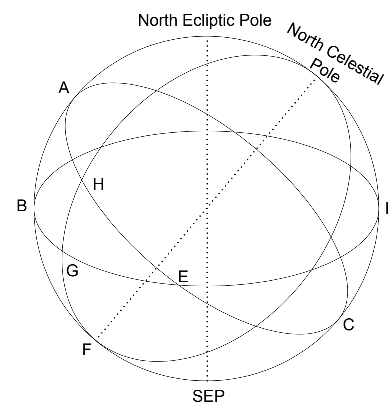
PTOLEMY

DAY 13

HOW TO FIND ARCS OF MERIDIAN INTERCEPTED BETWEEN THE ECLIPTIC AND EQUATOR GIVEN THE ARC FROM AN EQUINOX.

We can now use our Spherical Menelaos Theorems and our Table of Chords to determine the values of other arcs intercepted between the ecliptic and the equator at places other than 90° from the spring tropic (i.e. the point of intersection of the equator and the ecliptic).

For example let arc EG of the ecliptic = 30°



Given: Ptolemy's diagram, arc EG = 30°
EHA is celestial Equator
ABCD is circle through both sets of poles
B = winter tropic
D = summer tropic
E = spring equinox
F = south pole
arc FGH is the meridian through G

Find: Arc GH in degrees.

From the second spherical lemma:

cd2AF : cd2AB = (cd2FH : cd2GH) c (cd2EG : cd2BE)

but arcAF is 90°, arcAB is 23½°, FH is 90°, GH is sought, EG is 30°, BE is 90°.

So, doubling all those known arcs, and looking up their chords on our Table, and placing them in the proportion, we have:

$120^P : 48^P\ 31'\ 55'' = (120^P : \text{cd2GH})\ c\ (60^P : 120^P)$

Compounding both sides with the inverse of the last ratio, we have

(120 : 48 31' 55'') c (120 : 60) = 120 : cd2GH

Compounding the two ratios on the left, we have:

$120 \cdot 120 : (48\ 31'\ 55'')(60) = 120 : \text{cd2GH}$

Then we form the products of the means and of the extremes, and divide by 120 to find the cd2GH, and

cd2GH = 24 15’ 57’’

Using the Table of chords and arcs (and some interpolation of sixtieths), we find:

2GH = 23° 19’ 59’’

so arcGH = 11° 39’ 59.5’’

Q.E.I.

RINSE AND REPEAT.

Repeating this for other values of arc EG, we can produce a Table of Obliquity, which correlates various arcs of the ecliptic (from the spring equinox) with the lengths of the arcs of meridian cut off at that place between the ecliptic and the equator. Ptolemy gives just such a table in Chapter 15 of Book 1 of his *Almagest*.

We will refer to this table when we come to the material in *Almagest* Book 2, Chapters 1-4, when learning how to determine what the longest day of the year is in a given latitude, and similar information peculiar to given locations on earth.

PTOLEMY’S TABLE OF OBLIQUITY

Arcs of the Ecliptic from the Equinox in Degrees	Intercepted Arcs of the Meridian in Degrees, Minutes, Seconds		
1	0	24	16
2	0	48	31
3	1	12	46
4	1	37	0
5	2	1	12
6	2	25	22
7	2	49	30
8	3	13	35
9	3	37	37
10	4	1	38
11	4	25	32

12	4	49	24
13	5	13	11
14	5	36	53
15	6	0	31
16	6	24	1
17	6	47	26
18	7	10	45
19	7	33	57
20	7	57	3
21	8	20	0
22	8	42	50
23	9	5	32
24	9	28	5
25	9	50	29
26	10	12	46
27	10	34	57
28	10	56	44
29	11	18	25
30	11	39	59
31	12	1	20
32	12	22	30
33	12	43	28
34	13	4	14
35	13	24	47
36	13	45	6
37	14	5	11
38	14	25	2
39	14	44	39
40	15	4	4
41	15	23	10
42	15	42	2
43	16	0	38
44	16	18	58
45	16	37	20
46	16	54	47
47	17	12	16
48	17	29	27
49	17	46	20
50	18	2	53
51	18	19	15
52	18	35	5
53	18	50	41
54	19	5	57
55	19	20	56
56	19	35	28
57	19	49	42

58	20	3	31
59	20	17	4
60	20	30	9
61	20	42	58
62	20	55	24
63	21	7	21
64	21	18	58
65	21	30	11
66	21	41	0
67	21	51	25
68	22	1	25
69	22	11	11
70	22	20	11
71	22	28	57
72	22	37	17
73	22	45	11
74	22	52	39
75	22	59	41
76	23	6	17
77	23	12	27
78	23	18	11
79	23	23	28
80	23	28	16
81	23	32	30
82	23	36	35
83	23	40	2
84	23	43	2
85	23	45	34
86	23	47	39
87	23	49	16
88	23	50	25
89	23	51	6
90	23	51	20

This last value, of course, is half "the arc between the tropics," which we observed in Day 9. There, we said that the arc between the tropics was about 47° 40'. Half of that would be 23° + 30' + 20', or 23° 50'. Apparently, Ptolemy had a more precise value than that!

PTOLEMY

DAY 14

PROPERTIES OF TERRESTRIAL PARALLELS

In Chapters 1 through 6 of Book 2 of his *Almagest*, Ptolemy investigates the properties of terrestrial parallels in light of his astronomical model. Today we will follow some of those considerations of his.

WHERE WE ARE ON EARTH. In Chapter 1 of Book 2 of his *Almagest*, Ptolemy says how we can determine which hemisphere we are in, the northern or the southern. I know that I am in the northern hemisphere because I can see Polaris and the constellations near it. Another property of the northern hemisphere is that on an equinox, when the sun is on the celestial equator, the noon-shadows all fall to the north, i.e. the sun is in the southern half of our sky. (The opposite would be true in the southern hemisphere.)

He also says that all the people he has ever heard of live on one half of the Northern hemisphere (the world's population is larger and more broadly distributed today, of course, and even in his day there were people in the southern hemisphere, though he had little or no knowledge of them). To argue for this he appeals to events which can be viewed by people living very far apart, and which they could know were simultaneous even without having nearly-instantaneous telecommunications (like cell phones or email) to talk to each other. For example, if an astronomer in the far East is observing the moon while it is being eclipsed by the earth on a certain date, he might record the moon's longitude in his sky at the exact moment when earth's shadow reached the center of the moon, or when it first perfectly covers the moon. Meanwhile, an astronomer in the West would be observing the same thing, and those events would be more or less simultaneous for him. The fact that they can simultaneously observe certain events in the sky proves that they themselves live less than 180° apart in longitude (with the whole known human population in between) on the Earth. If the known human population had covered more than 180° of longitude on the Earth, then there would be records of total lunar eclipses which were observed by some of those people on certain dates whereas others would not have observed them, or would have seen only the ending of the eclipse. As that did not happen, or not among the peoples known to Ptolemy, he concluded that they all lived in one half of the northern hemisphere.

This sort of question is not merely a matter of geography for Ptolemy. In many ways, it is the first business of the astronomer, his work closest to home: to determine the heaven-observing properties of different locations here on dear old Earth. Ptolemy will next study the astronomically relevant properties of different terrestrial latitudes, namely:

a. The height of the pole above the horizon.
b. The distance of the zenith from the equator on the meridian.
c. Whether the sun ever reaches the zenith, and if so, how often.
d. The noonshadow-to-stick ratio on equinoxes and solstices.
e. The lengths of the longest and shortest days and nights.

FIRST STUDY OF DIFFERENT LATITUDES, USING SPHERICAL PROOFS.

In Chapters 2 and 3 of Book 2 of his *Almagest*, Ptolemy considers three different pieces of information about a latitude:

1. The arc on the horizon EG, where E is the place that the Equator cuts the horizon (always) and G is the place where the winter solstice (on the ecliptic) rises (always). So the sun rises at G on the shortest day of the year.

2. The height of the pole above the horizon (or the depth of the opposite pole beneath the horizon), arc BF.

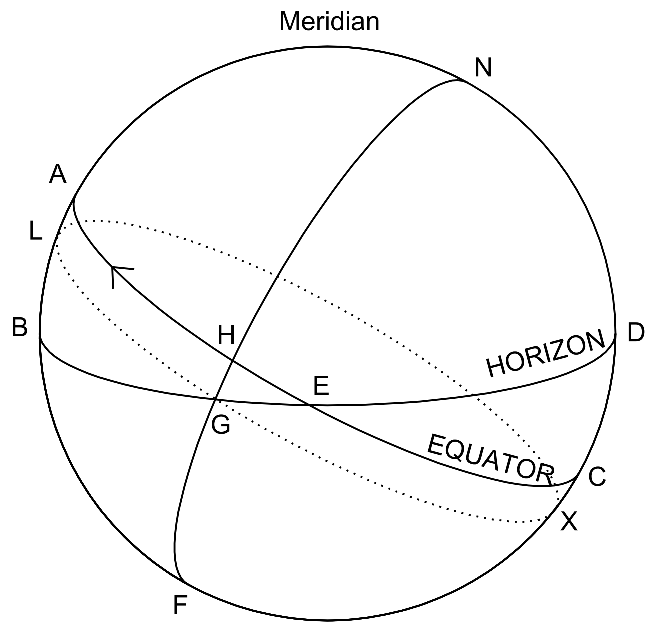
3. The length of the longest/shortest day of the year.

He shows the following things:

(a) Given the length of the longest day of the year in your latitude, you can determine the length of arc EG on your horizon (he does this in Ch.2).

(b) Given the length of arc EG on your horizon, you can determine the height of the pole where you are (he does this in Ch.3), i.e. you can find arc BF or DN. It follows that you can also determine these two pieces of information just from knowing the length of the longest day where you live.

(c) Given the height of the pole where you are, you can determine the length of the longest day of the year, and you can determine the length of arc EG on your horizon. (Ch.3).



LG = Half the shortest day

GX = Half the longest night

The last is the most interesting, since it allows us, without traveling, to determine the properties of latitudes to which we have never been, while sitting in our armchair by the fire.

The place he uses as an example for his data is Rhodes (an island off the southwestern tip of Asia Minor, located in the eastern Aegean Sea—it was an important economic center in the ancient world). Let's go through his examples, and then do an exercise of our own.

Given: Length of longest day = 14½ equatorial hours
Find: Arc EG on the horizon

Incidentally, an "**equatorial hour**" is one twenty-fourth of the equator, or the time it takes for one twenty-fourth of its arc to rise above our horizon. The motion of the sphere of fixed stars is exactly regular, so far as Ptolemy is concerned, whereas the sun has an additional motion besides this daily motion of the heaven, and it in fact speeds up and slows down in that additional motion, and because of this irregularity, it is better to define time-units by a more regular motion, such as that of the celestial sphere itself. An equatorial hour differs a bit from a solar hour, thanks to the sun's orbit around us (or our orbit around it). The equatorial hour is a 24th of the time for the sphere of fixed stars to rotate around us once.

The first thing he needs, before he can use the Spherical Menelaos Lemma to find arc EG, is to determine what 2arcHA is. (FGH is a celestial meridian, namely the one passing through the winter Tropic, G. Arc HA is the arc of equator cut off by this "tropical meridian" when the tropic is on our horizon BE.) His reasoning depends on seeing that any given arc on the celestial equator will take a specific amount of time to pass through our meridian.

How long will it take 360° to pass through our meridian? 24 equatorial hours.
How long will it take 180° to pass through our meridian? 12 equatorial hours.
Conversely,
How much arc of the equator passes through our meridian in 6 hours? 90°.
How much arc of the equator passes through our meridian in 3 hours? 45°.
So now we can translate equatorial time into equatorial arcs.

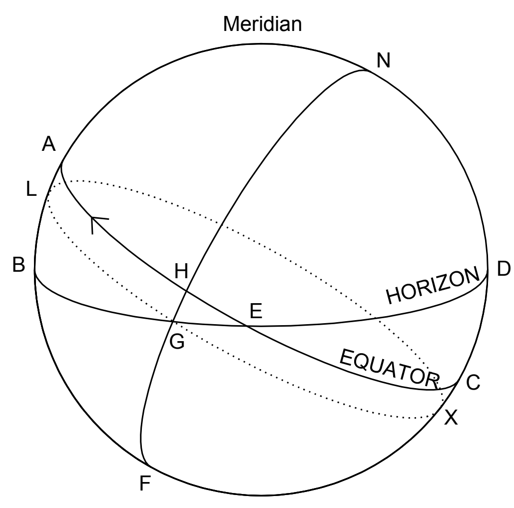
Now, since the length of the longest day is given as 14 ½ equatorial hours, therefore the length of the shortest day is 9 ½ equatorial hours. Therefore half the length of the shortest day is 4 ¾ equatorial hours.

And since H goes to A on the meridian in the same time that G goes to L, and G goes to L in half the length of the shortest day, therefore H goes to A in 4 ¾ equatorial hours.

Therefore arc HA : 360° = 4 ¾ hours : 24 hours

Solving for arc HA gives us that

arc HA = 71.25°
i.e. arc HA = 71° 15'
thus 2 (arc HA) = 142° 30'



LG = Half the shortest day
GX = Half the longest night

Before we can use the Spherical Menelaos we also need to know 2arcFG. Well, we know that

arcFG = arcFH – arcGH

so arcFG = 90° – 23½°

since arcGH is the arc of the great circle through the poles which is intercepted between the winter solstice (on the ecliptic) and the equator.

Hence arcFG = 66½°
so 2arcFG = 132° 17' 20'' (to be more exact than 133°)

Now we can proceed (I will not spell out all the minutes and seconds):

cd2HA : cd2AE = (cd2FH : cd2FG) c (cd2BG : cd2EB)[1]

i.e. cd142° : cd180° = (cd180° : cd132°) c (cd2BG : cd180°)

Using our table of chords:

$113^P : 120^P = (120^P : 109^P)$ c (cd2BG : 120^P)

So, compounding by multiplying the numbers,

$113^P : 120^P$ = (120·cd2BG) : (109)(120)

And equating the products of means and of extremes:

(113)(109)(120) = (120)(120)(cd2BG)

so (113)(109) ÷ 120 = cd2BG

i.e. cd2BG = 103^P

or cd2BG = 103^P 55' 26'' (to be more exact, like Ptolemy)

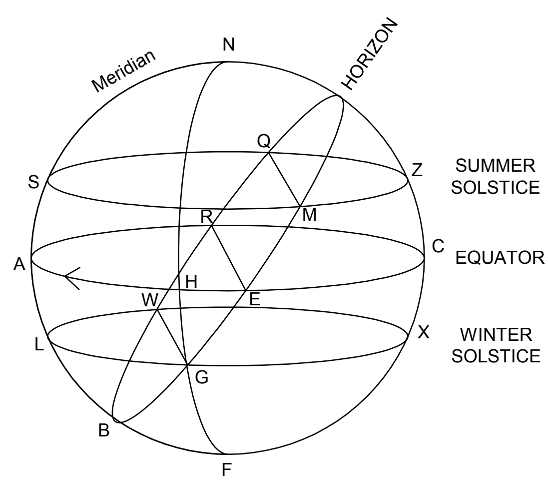
Looking at our table of chords and arcs, we get

2BG = 120°

so arcBG = 60°

so arcEB – BG = 90° – 60° = 30° = arc EG *Q.E.I.*

So, if you tell me the length of your longest day (or shortest, or whatever), I can tell you where on your horizon the sun will rise on the shortest day of the year.



LG = Half the shortest day
GX = Half the longest night

[1] Inverse of the footnote on p 28.

GIVEN THE LENGTH OF THE LONGEST DAY OF THE YEAR IN SOME LOCATION, TO FIND THE HEIGHT OF THE POLE THERE.

Given:	length of longest day = 14 ½ equatorial hours (hence arc EG = 30°, as shown above; hence arc BG = 60°) (hence, also, 2arcHA = 142° 30', as above) (hence, also, 2arcEH = 37° 30', by subtraction from 180°)
Find:	arcBF (the height, or depth, of the pole)

Again, I will leave out all the minutes and seconds to keep things simple.

Well, cd2EH : cd2HA = (cd2EG :cd2BG) c (cd2BF : cd2AF)

i.e. cd37° : cd142° = (cd60° : cd120°) c (cd2BF : 180°)

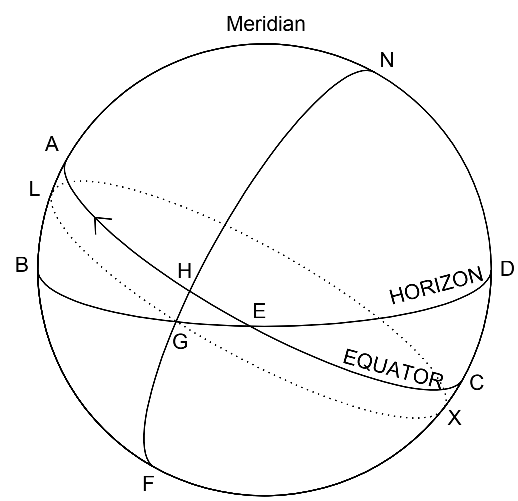
so $38^P : 113^P = (60^P : 103^P)$ c (cd2BF : 120^P)

thus cd2BF = 70^P

so 2arcBF = 72°

so arcBF = 36°

Q.E.I.



LG = Half the shortest day

GX = Half the longest night

GIVEN THE HEIGHT OF THE POLE IN SOME REGION, TO FIND THE LENGTH OF THE LONGEST AND SHORTEST DAYS OF THE YEAR IN THAT SAME PLACE.

Given: The height of the pole = arc BF = 36°
(hence 2arcBF = 72°)
(hence, too, arcAB = 54°, so 2arcAB = 108°)
(also, arcFG = arcFH – arcGH = 90° – 23° = 66°, so 2arcFG = 132°)
(and 2arcGH = 47°, the arc between the tropics)
(and 2arcAE = 180°)

Find: Difference between the longest (or shortest) day and the equinoctial day (in equatorial hours).

(Again, I'll use rounded numbers to keep things simple.)

Well, cd2BF : cd2AB = (cd2FG : cd2GH) c (cd2EH : cd2AE)

i.e. cd72° : cd108° = (cd132° : cd47°) c (cd2EH : cd180°)

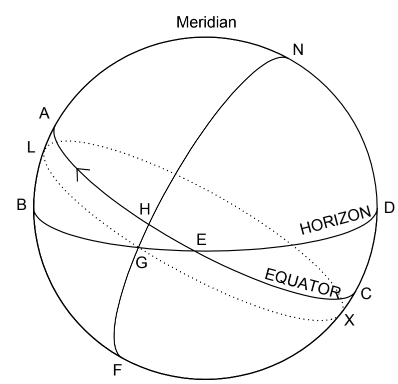
so 70 : 97 = (109 : 48) c (cd2EH : 120)

so (70 : 97) c (48 : 109) = cd2EH : 120

so 3360 : 10573 = cd2EH : 120

so cd2EH = 38

so 2arcEH = 37½ °

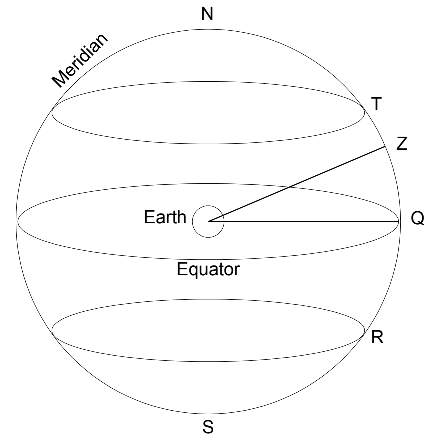


But it takes the same time for G to go to L on our meridian as it takes H to go to A on our meridian, since FGH is a celestial meridian, spinning about the celestial poles. But for the sun to go from G to L takes half the shortest day; therefore the time for H to go to A is half the shortest day. But arc EA is 90°, so the time for that arc is half the equatorial day. Hence 2arcEH is the arc whose time corresponds to the difference between the shortest day and the equinoctial day.

Since 2arcEH = 37½ °
thus difference between shortest & equinoctial : 24 = 37½ : 360
so diffc. btwn. shortest & equinoctial days = 2.5 hours

Hence the shortest day of the year = 12 – 2.5 = 9 ½ equatorial hours
and the longest day of the year = 12 + 2.5 = 14 ½ equatorial hours
Q.E.I.

DOES THE SUN PASS THROUGH THE ZENITH IN YOUR LATITUDE? IF SO, WHEN?



Ptolemy asks this question in Chapter 4 of Book 2 of his *Almagest*.

If your latitude on earth is more than 23½° north or south of the Earth's equator, the sun never reaches your zenith, Z.

T is where the path of the summer tropic cuts your meridian NTZQRS.

Q is where the equator cuts your meridian.

R is where the path of the winter tropic cuts your meridian.

So if your zenith Z is on arc TR, the sun is at your zenith at least once a year (twice a year if Z is *between* T and R, and not on one of them).

WHEN does this happen?

You can measure arc ZQ in your sky. But this is a piece of your meridian cut off between the ecliptic and the equator, on a day of the year that the sun is at Z (since the sun is always on the ecliptic). Hence arc ZQ is an entry in our Table of Obliquity (Day 13, pp.91-93). Just look up the value of arc ZQ (as measured or observed in your sky with his Meridian Machine of Ch.12) in the "Meridian" column of the Table of Obliquity, and see how much arc of ecliptic is between Z and the spring equinox. But given the location of the Sun in degrees from the spring equinox, one can say what time of year it is, or on what day of the year the sun will be at Z. For

1 full year : X amount of time from the last spring equinox

= 360° : number of degrees the Sun is from the spring equinox

So we can solve for X, which will tell us on what day of the year the sun will be at our zenith.

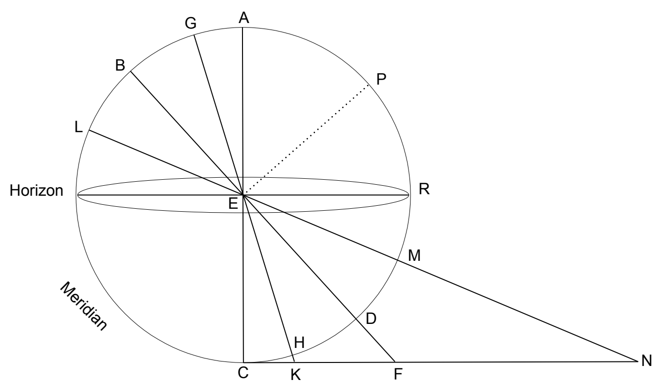
EXERCISE: Find the height of the pole above your horizon by observing where Polaris appears above your horizon using a simple device (e.g. a hinged pair of sticks with a nut that can tighten and hold the angle fixed, and a protractor to measure the angle between them). Then, using the techniques we learned, and given your latitude by the height of the pole which you observed, calculate the length of the longest day of the year (in equatorial hours) where you live, and say whether the sun reaches our zenith or not. Check your answers against values which you can look up on the internet, and see how close you come.

PTOLEMY

DAY 15

MORE PROPERTIES OF TERRESTRIAL PARALLELS

Ptolemy continues his investigation of the properties of different latitudes on earth in Chapters 5 and 6 of Book 2 of his *Almagest*. In Chapter 5, he considers this problem: Given some latitude on earth, find the ratios of the noon-shadows cast by the same stick at that latitude on the summer solstice, the winter solstice, and the equinoxes. We could just wait for those times of year and measure the three shadows (the ones on the equinoxes should be equal), but we don't have to wait. We can figure it out right now. We'll use Ptolemy's example of the latitude of Rhodes to illustrate the method.



Given: Latitude = 36° (Rhodes)
Find: Ratios of the summer, winter, and equinox noon-shadows

E is Earth
P is Polaris
A is Zenith
B is where the equator passes through the meridian
G is summer tropic
L is winter tropic

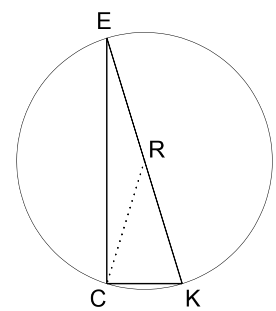
Draw lines through LEN, BEF, GEK through to the tangent at C, and the triangle CEN now is similar to a stick CE casting shadows proportional to CK, CF, CN when the sun is at the summer tropic, equinox, and winter tropic respectively. (The diagram is like an enormous magnification of a similar scenario going on at E; OR it simply *is* the diagram of the stick, imposed on a meridian circle which is a miniature of the real celestial meridian.)

Now the arc between the zenith & equator (on the meridian) is equal to the arc between the horizon & the north pole (on the meridian), i.e.

	arcAB = arcPR = 36°	[the given latitude]
but	arcBG = 23° 51’ 20’’	[½ the arc between tropics]
so	arcAG = 12° 8’ 40’’ = arcCH	
and	arcAL = arcAB + arcBL = 36° + 23° 51’ 20’’ = 59° 51’ 20’’ = arcCM	

so ∠CEK = arcCH = 12° 8’ 40’’
and ∠CEF = arcCD = arcAB = 36°
and ∠CEN = arcCM = 59° 51’ 20’’

So now, taking CA as 120, or CE as 60, what are the shadows CK, CF, CN?

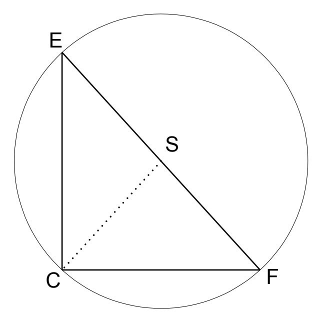


Well, ∠CEK = 12° 8’ 40’’
so ∠CRK = 24° 17’ 20’’
so CK = 25^P 14’ 43’’ where EK = 120
so CE = 117^P 18’ 51’’ (chord of the supplement of ∠CRK)

So if we decide to call CE 60 now, then

117^P 18’ 51’’ : 25^P 14’ 43’’ = 60^P : CK

so CK = 12^P 54’ 42’’ when CE = 60

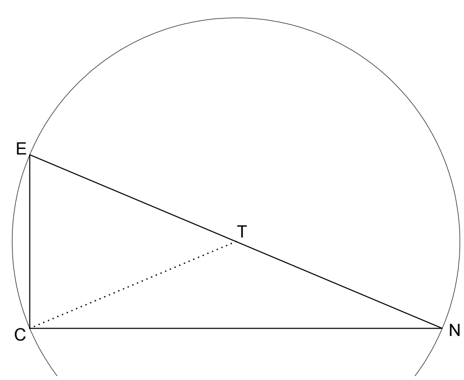


Now ∠CEF = 36° (since arc CD = 36°)
so ∠CSF = 72°
so CF = 70^P 32’ 3’’ where EF = 120
so CE = 97^P 4’ 56’’ (chord of the supplement of ∠CSF)

So if we decide to call CE 60 now, then

97^P 4’ 56’’ : 70^P 32’ 3’’ = 60^P : CF

so CF = 43^P 35’ 33’’ when CE = 60



And ∠CEN = 59° 51’ 20’’
so ∠CTN = 119° 42’ 40’’
so CN = 103^P 46’ 16’’ where EN = 120
so CE = 60^P 15’ 42’’ where EN = 120

So if we decide to call CE 60 now, then

60^P 15’ 42’’ : 103^P 46’ 16’’ = 60^P : CN

so CN = 103^P 19’ 14’’ when CE = 60

In Chapter 6 of Book 2 of his *Almagest*, Ptolemy provides a kind of table or almanac of the properties of various latitudes on earth, as determined by the foregoing methods. We will not discuss them all here, but only some of the more interesting ones, namely Parallels i, vii, x, xi, xxxiii, xxxiv, xxxix

We have seen that given the length of the longest day of the year in some spot on earth, we can find the height of the pole in that place, i.e. the latitude (see Day 14). Therefore we can find the latitude of places where:

Longest day = 12 hours
Longest day = 12 ¼ hours
Longest day = 12 ¾ hours
Longest day = 13 hours etc.

And so Ptolemy divides terrestrial latitudes by increments of ¼-hour increases in the length of the longest day of the year from the latitude where the longest day is 12 hours (the equator) to the place where the longest day is 18 hours. From 18 to 20 hours, he divides by ½-day increments, and from 20 to 24 hours he divides by whole-day increments, and then after that he jumps by much longer amounts of time (since at the North Pole the longest day of the year, in a sense, is 6 months!).

Now a little point about Ptolemy’s vocabulary.

A “**gnomon**” is what Ptolemy calls a stick in the ground used to cast a shadow and tell us things about the Sun. The word “gnomon” comes from the word for knowledge. So a “gnomon” is an index, an indicator, a pointer. And the stick in the ground points to the sun if there is no shadow at all, or the line joining the tip of the shadow to the top of the gnomon points directly to the Sun (which we can’t look directly at, or not safely).

Also, “skia” is Greek for shadow, or shade. So “**amphiscian**” means ambi-shadowed, i.e. a place where noon-shadows can point to the north and also to the south at different times of year (due north and due south at opposite solstices). “**Heteroscian**” means “one-or-the-other-shadowed”, or preferentially-shadowed, e.g. a place where noon-shadows can only point north, never south (or vice versa). “**Periscian**” means “round-about-shadowed,” i.e. a place where the shadows can point in all directions. Let’s consider these properties in specific parallels.

SPECIFIC PARALLELS

[i] The parallel under the celestial equator (i.e. earth's equator) has the following properties:

1) It's the only place where every day equals every night.

2) It is "amphiscian," i.e. the noon-shadows fall north in winter, south in summer, so the shadows fall both ways.

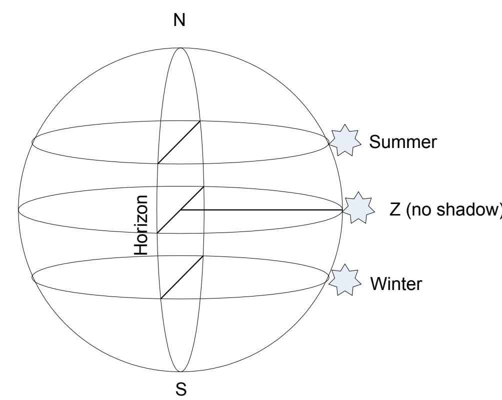
3) The sun passes through the zenith once at each equinox.

4) The summer solstice noon-shadow = winter solstice noon-shadow (if you use the same gnomon).

5) The poles are on the horizon.

6) No star traces out a full circle in the sky, but rather all trace out 180°. There are no "always visible" stars, i.e. all stars rise and set.

7) The weather is warm, since the sun's path never gets very low in the sky.

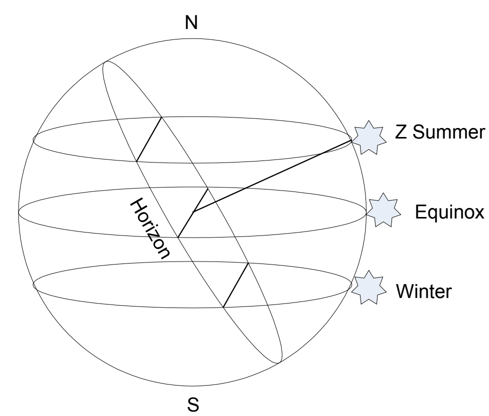


[vii] This parallel is the terrestrial Tropic of Cancer, a circle traced out on the surface of the Earth (once every 24 hours) by a line joining Earth's center to the summer tropic on the ecliptic. Here the longest day is 13 ½ hours. This latitude is 23° 51' north of Earth's equator, and has the following properties:

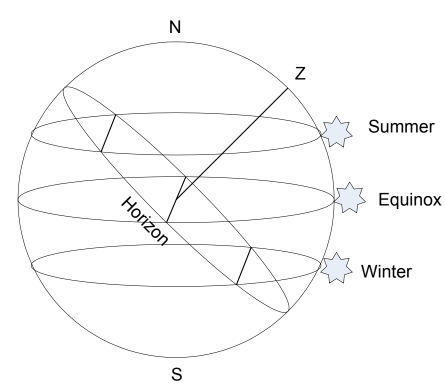
1) This is the first "heteroscian" parallel: noon-shadows fall north all year, but at the summer solstice, when the sun is at the zenith, there is no noon shadow.

2) The sun passes through the zenith only once a year (at the summer solstice).

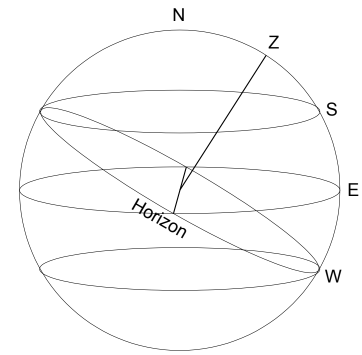
3) The winter solstice noon-shadow is greater than the equinox noon-shadow.



[x] Latitude 33° 18' north of the equator. The longest day here is 14¼ hours. The summer solstice noon-shadow is less than the equinox noon-shadow which is less than the winter solstice noon shadow. This latitude is clearly heteroscian, since the noon-shadows are cast to the north year round.



[xi] Latitude 36° north. The longest day is 14 ½ hours.
Again: summer solstice noon-shadow < equinox noon-shadow < winter noon-shadow.



[xxxiii] Latitude 66° 8' 40'' north. The longest day is 24 hours. Anywhere closer to the equator will not have a day that long, and anywhere north will have a day that is longer than 24 hours. So this is the Arctic Circle. We can determine this latitude which is the first to have a 24 hour day, since that will be the place where the sun's orbit at the summer solstice just touches the horizon, so that on that day, in that place, the sun is above the horizon for one full rotation, and then begins its southern descent again, so that it will be partly below the horizon the next day. But where will that be? It will be in that latitude where one's horizon intersects the sun's paths on the solstice days. But those paths each lie 23° 52' 20'' away from the equator, one that far north of it, the other that far south of it, as we determined in Day 9. So if Z is the zenith at such a latitude, and S is where the sun cuts the meridian at the summer solstice, and E is where the equator cuts the meridian, and W is where the sun cuts the meridian at the winter solstice, then

Arc EZ = arcWZ – arcWE = 90° – 23° 52' 20'' = 66° 8' 40''.

But the angle between one's zenith (Z) and the equator (E) is one's latitude.

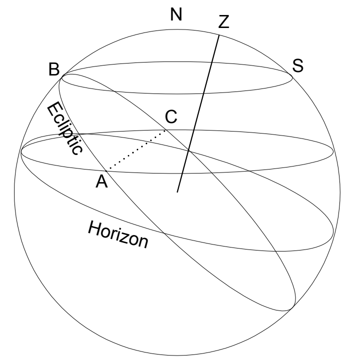
When the sun is at S, there is a 24 hour day.
When the sun is at W, there is a 24 hour night.

1) This is the first "periscian" parallel, i.e. where the shadows can fall in all directions (360°), namely on the summer solstice, when the shadow of the gnomon goes around in one full circle.

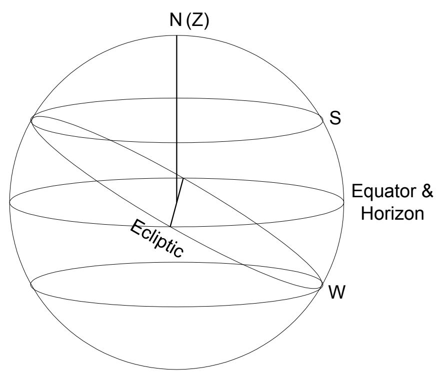
2) This is the first parallel from which one can see the whole celestial parallel through the summer tropic, and none of the parallel through the winter tropic, in the heavens.

3) At this parallel, the ecliptic coincides with the horizon once every 24 hours.

[xxxiv] Latitude 67° north (very nearly). Here the longest day is one month. Here, arc ABC of the ecliptic never sets, and so as long as the sun is on that part of the ecliptic, the sun never sets.



[xxxix] Latitude 90° north, i.e. the terrestrial North Pole. The horizon always coincides with the equator. Here there is a 6 month day and then a 6 month night. The northern 180° of the ecliptic never sets, and the southern 180° of it is always below the horizon. All the stars go about the observer in circles parallel to the horizon.



PTOLEMY

DAY 16

THE BASICS OF SOLAR THEORY: THE SOLAR PERIOD AND ITS REGULAR AND IRREGULAR MOVEMENT

We are now in Book 3 of the *Almagest*. We've spent a lot of time on Earth. Now it's time to begin studying the things in the heavens themselves, not just the appearances they cause at various latitudes on our world. Following Ptolemy's order, we will begin with the sun. There are several reasons to do this. First of all, it is in many ways the easiest thing to observe. Second, it is somehow fundamental to the movements of the other planets, as we shall see—they all exhibit patterns of behavior and cycles that are clearly connected to the sun. (This odd fact, which will strike us more and more as we go through Ptolemy, should make us begin to wonder whether we got it quite right when we made everything, including the sun, go around the Earth.) After we have considered the basics of Ptolemaic solar theory, we will skip the lunar theory (in the interests of brevity and simplicity) and study the planets, focusing on one "inner" planet, Venus, and one "outer" planet, Saturn.

We have already seen that the sun has two motions: one by which it moves daily with the celestial sphere, rising and setting in circles parallel to the celestial equator, but another one by which it moves in the opposite direction on the ecliptic, which is at an angle of about 23.5° to the celestial equator, and which it completes in a year. Strictly speaking, then, the sun does not make parallel circles, but more like a spiral. As it crawls backward along the ecliptic, it is also being spun around much more rapidly by another sphere which moves just like the celestial sphere, and the sun's path compounded of these two motions is a giant coil or helix going from the circle of the summer tropic down through the equator and to the circle of the winter tropic, without about 365 passes in it.

Our first question about the sun's proper movement on the ecliptic will be about the *time* it takes, or its "**period**." How long does it take the sun to go from one point on the ecliptic back to that same point again once? For example, from spring equinox back to spring equinox? Of course it takes about a year, or around 365 days. But we want to get a nice *exact* figure, as precise as we can, so that we can chop this up and say exactly how far the sun moves (in degrees, now, not in miles or kilometers) on its annual orbit even in small amounts of time, e.g. in a single hour or in five minutes. That sort of information will become handy as we proceed to the study of the planets. And, as we shall see, certain key advances in astronomy become possible only with great precision and accuracy in our observations. That is seen most plainly in Kepler, but we shall begin to see it even in Ptolemy.

THE PRECISE LENGTH OF THE YEAR
i.e. OF THE PERIOD OF THE SUN'S MOTION ON THE ECLIPTIC

Ptolemy sets about determining the period of the sun's motion on the ecliptic in Chapter 1 of Book 3 of his *Almagest*. He refers to a certain gentleman named "Hipparchus," another astronomer who lived about 260 years before Ptolemy (see *Almagest* Book 7 Ch.1). Hipparchus also wished to determine the solar period, but in doing so he ran into a problem, which can be summed up like this:

[A] If we measure the time from when the sun cuts (e.g.) the northernmost point on our meridian or horizon (namely at summer solstice) to when it comes back to that point, then

A YEAR < 365 ¼ days

[B] If we measure the time from when the sun is at one place among the fixed stars to the next time it is there,

A YEAR > 365 ¼ days

That is an odd discrepancy! To understand it, we need to grasp what the two methods are for measuring the period. The second one, [B], is easier to understand. We take observations of the Sun when it is setting, and look at the opposite point on our horizon, and see where that point falls in the zodiac (using the rising stars there to guide us). Then we look at our star charts, and see what point in the zodiac is exactly 180° away from the point we observed, and that is the point where the Sun is. We then wait for the Sun to move through all the stars and come back to that same spot in the zodiac, about a year later, keeping count of the days. Now it is hard to say in what part of a given day the Sun is exactly in some spot in the zodiac—so what's with the "¼ day" business? One way to get more precise than to the nearest day is to count the days not during a single return to the same spot in the zodiac, but the days taken for the sun to return to the same spot 50 times, or 100 times, or more. If we divide the total number of days by the number of times the sun went round, we will have a more precise figure for its period than just to the nearest day.

The first method, method [A], is from one solstice day back to the same solstice day, e.g. from summer solstice to summer solstice. How do we know that the sun is at the summer solstice? If we just say "because it is the longest day of the year," then we will get a figure for the period which is only to the nearest day. We need more precision than that. One way is to take a more accurate observation of a solstice (or equinox, for that matter), by locating the furthest point north of the equator at which the sun has ever been observed. The sun is only at the northern tropic for an instant in time, climbing up to it and passing right through it and coming south again. So if we observe carefully on the day of the summer solstice, we might see that at some point during the day it is *not quite* as far north of the equator as it has been observed in the past, and later in the day it *is* that far north. Later still, it is no longer that far north. We have witnessed the exact moment of the solstice! If we can get a fairly accurate observation that way, then we can get a value for the period of the sun (or the length of the year) which is more precise than to the nearest day. And we can also use the same technique as in method [A], measuring the time not between one solstice and the next similar one, but between one solstice and a similar one 50 cycles later.

At any rate, measuring the length of the solar period in these two different ways, we get two distinct values! How did Hipparchus explain this?

Hipparchus explained this by assuming there is a very slow eastward motion of the sphere of fixed stars contrary to its prime movement, and about the poles of the ecliptic. (This is the same as the "precession of the equinoxes," or the slow wobble of earth's axis—more on this later.) The sphere of fixed stars, like the sun, has two movements, and so two sets of poles—one around which it moves with the daily motion, another around which it moves with a much slower motion, in the opposite direction. So the solar year is slightly longer, measured against the fixed stars, because they are also moving with a "motion of the other" (i.e. eastward), very slightly. It is as if the sun were a runner on the inside track, and the stars were a runner on an outside track, and the stars were moving in the same direction as the sun, but extremely slowly. If we don't notice that the stars are running, we can think that the sun is back to its same spot just because it is caught up with the same star once again. But that star has moved a tiny bit on the track! And that means that the sun has actually gone one full circle *plus a tiny bit*.

Ptolemy sides with Hipparchus and chooses to define the solar period (or year) as [A], the time for the sun to make one full circuit around the ecliptic (although once it returns to that point, it is not in exactly the same position relative to the fixed stars). So, for instance, it is the time from equinox back to the same equinox.

Why choose this one? Because that is proper to the sun itself, relative only to us (who are fixed). Defining with reference to the fixed stars is arbitrary and extrinsic, because that sphere itself has a motion contrary to the prime motion. One might as well take Saturn, and say the time it takes for the Sun to be again at the same longitude with Saturn is "one solar year." That is obviously arbitrary.

So we must pick what we mean by saying that the sun is "again at the same point." Not with reference to fixed stars. Not with reference to a planet. Then what? Well, when it is again at the *northernmost point* of its orbit; or the *southernmost point*; or exactly in between (at an equinox).

How do we accurately find the length of the year thus defined?

This "year," as we have defined it, falls short of the ¼ day (above the 365 days) by so slight an amount, no difference is easily seen in one year from the 365 ¼ days. And we cannot get an accurate and precise *observation* of how far it falls short of the ¼ day, because the exact instant in which the sun is on the celestial equator is tough to pinpoint, and it might happen when the sun is below our horizon.

• Thus we find the difference by multiplying it many times (i.e. over many years) and dividing. Similarly, if you have a sheet of paper, and you are wondering how thick it is, and you know that it is less than a millimeter, but you can't say by how much with any precision by measuring the thickness of a single sheet, you can stack up 500 sheets and (assuming there is no air between them, and all the sheets are exactly equally thick) measure the stack in millimeters and divide by 500, and you will get a precise value for the thickness of one sheet! For instance, if the stack of paper is 30 millimeters thick to the nearest millimeter, then each sheet is .06 millimeters thick.

• Likewise we have a pile of identical years. And while we can't get a very precise value for one of them just by observing the time from a spring equinox back to a spring equinox, thanks to observations of many astronomers gathered by Hipparchus we know that

300 years in a row = [(365.25)300 – 1] = 109574 days

so 1 year = 109574 ÷ 300 = 365.246 days
or 1 year = 365^{D} + 14' + 48''
i.e. 1 year = 365 days + 14 sixtieths of a day + 48 3600^{ths} of a day
i.e. 1 year = 365 days 5 hours 55 minutes 12 seconds

That gives us the length of the year to the nearest second. Pretty good.

But now how do we calculate solar movement on the ecliptic in a given time? How do we know how far it goes in a single day, or hour, or minute?

Short answer: We assume that the sun's motion is perfectly uniform in angular velocity on its own perfectly circular orbit. No proof is given for this—it is part of the world-view of Ptolemy. This bears on the "Astronomer's Axiom" which is discussed below.

If we assume that the sun's motion is totally uniform, then obviously since we know how long it takes to go 360°, we also can determine how long it takes to go any given number of degrees, or, conversely, given a time, we can say how many degrees it moves on its circle. So Ptolemy builds up tables of the sun's uniform motion.

For example, since the sun goes 360° of ecliptic in 365.246 days, then if we divide by 365.246, we get how many degrees the sun goes in 1 day,

i.e. .985635278° in one DAY
or, sexagesimally,
$[59/60 + 8/60^2 + 17/60^3 + 13/60^4 + 12/60^5 + 1/60^6]°$ in one DAY.

Now multiply that by 30 to get how far it goes in one Egyptian MONTH (which is exactly 30 days), and by 365 to get how far it goes in one Egyptian YEAR (which is exactly 365 days), and by (365 × 18) to get how far it goes in 18 Egyptian years:

Sun goes 29.56911° in 1 Egyptian MONTH
i.e. $[29 + 34/60 + 8/60^2 + 36/60^3 + 36/60^4 + 15/60^5 + 30/60^6]°$

Sun goes 359.757505° in 1 Egyptian YEAR
i.e. $[359 + 45/60 + 24/60^2 + 45/60^3 + 21/60^4 + 8/60^5 + 35/60^6]°$

Sun goes 6475.63509° in 18 Egyptian YEARS i.e. 17 full circles plus 355.63509°

i.e. 17 full circles plus $[355 + 37/60 + 25/60^2 + 36/60^3 + 20/60^4 + 34/60^5 + 30/60^6]°$

In Chapter 2 of Book 3 of his *Almagest*, Ptolemy produces three **Tables of the Sun's Regular Movement**.

TABLE 1 gives the surplus of the sun's mean motion over some number of complete circles in 18 Egyptian years, in 18 × 2 Egyptian years, in 18 × 3 Egyptian years, etc.

Note: the Egyptians had a year of 360 days, plus 5 "intercalary days" when they would stop working and drink beer. These 5 days were not on the calendar, since there were 12 months of 30 days each, which gave a year of 360 days, leaving 5 days extra.

TABLE 2 gives the surplus of the sun's motion over some number of complete circles in 1 Egyptian year, 2 Egyptian years, 3 Egyptian years, etc.

This table also gives the measure of the sun's motion in 1 hour (i.e. in the time it takes to go 1/24 of 1/365.246 of its full 360°), in 2 hours, etc.

TABLE 3 gives the measure of the sun's motion in 1 Egyptian month (30 days), in 2 months, etc. It also gives the measure of the sun's motion in 1 day (i.e. the time it takes to go 1/365.246 of its full 360°), in 2 days, etc.

THE (ANCIENT) ASTRONOMER'S AXIOM.

It is a foundational principle of science that the complex should be explained through the simple, as far as this is possible. There is a good deal of inductive support for thinking that nature behaves simply and follows relatively simple rules, and there is also some reason to think this. But what exactly counts as "simple"? That's the hard part. If we see planets looking like they do zig-zags in the sky, sometimes stopping, sometimes going this way, sometimes the opposite way, one "simple" understanding of this is that they are really doing just what they look like they're doing—they're stopping, going backwards, going forward again, stopping again, and so on. If we were content with that sort of thing, we would not get very far in our understanding of the world. We find, time and time again, that if we assume these sorts of behaviors are in fact just irregular *appearances* of underlying regularities, we turn out to be right somehow, and our understanding advances.

Ptolemy's understanding of "simple" is very simple. He assumes that planets move at perfectly uniform speeds on perfect circles—sometimes more than one circle. With that assumption about them, he sets out to explain all the appearances in the heavens as products of uniform motions around perfect circles. That is the whole purpose of his astronomy, one might say.

Like other first attempts to understand truth, this "axiom" is not wholly false. It is not an "axiom" in the sense of being self-evident, like "equals added to equals make equals." But, like "the earth sits still," or "heavier things fall faster," it is close to something self-evident, and it is partly true:

(a) The planets come so close to making perfect circles with uniform speed around the sun that people laugh when they see how little Kepler's orbit for Mars differs from a circle (even though it is one of the more elliptical orbits), and Newton often calls the planetary orbits "circles" and applies to them propositions which are true only about uniform motions in circles.

(b) Even according to modern physics, it is exactly true that there are mathematical uniformities, regular laws, underlying the non-uniformity of planetary motions.

ANOMALIES

Together with this axiom of the astronomer, Ptolemy also introduces the word "**anomaly**," which means any apparent speeding up and slowing down of a heavenly body. The word "anomaly" comes from the Greek word for "unlawful" or "lawlessness." A planetary anomaly is an apparent "lawlessness" (or "irregularity," an "unruly" behavior) in its motion, which must be explained as the product of "lawful" motion, i.e. uniform motion around some perfect circle. (Aristotle, who lived some centuries before Ptolemy, insisted on all the circles being concentric, for physical reasons; he thought the heavenly bodies all have natural motions around the center of the world and not to or from it. Ptolemy does not insist on that, partly because the appearances are not easily explained on such a hypothesis, and partly because he does not concern himself as much with trying to understand the physical nature or natural tendencies or motives of the celestial bodies.)

In keeping with the Astronomer's Axiom, Ptolemy says that *any anomaly is merely apparent, not a real speeding up or slowing down of the heavenly body, but only relative to an observer, because of the position of the circle that the body is on, relative to us*. So it is the job of the astronomer to separate out the "**mean**" motion, i.e. the uniform motion of a body, and the cause of the anomaly, i.e. some eccentricity or epicicyle (more on this later).

SOLAR ANOMALY

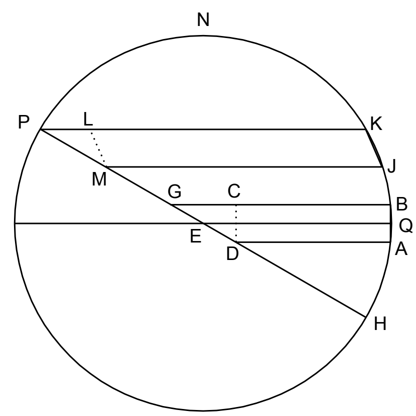
Getting back to the sun, it exhibits certain anomalies. In particular, it appears to move now faster, now slower, along the ecliptic. In *Almagest* 3.4, Ptolemy specifies that:

Time from Spring Equinox to Summer Tropic = 94.5 days

Time from Summer Tropic to Autumn Equinox = 92.5 days.

The numbers come from Hipparchus. They are *not* inferred from some fact of ordinary experience. One has to bother to observe the times between equinoxes and solstices carefully, preferably over many cycles, as explained above.

At any rate, Ptolemy cannot just assume that the sun in fact slows down and speeds up at different places in its orbit. The mere fact that it does so regularly, cyclically, taking the same time through any given part of its orbit, indicates some deeper regularity. No, he will not assume the appearance of irregularity means there is some real irregularity in the sun's motion. Instead, he will explain it by the eccentricity of the sun's circular path around us—in other words, the center of that circle is off the center of the universe where we are, so we are not viewing the sun's motion from the point around which it is moving uniformly. We will get into the details of this soon.



NOTE: The gross inequality in the seasons we all note, for example the daylight gets longer faster near the spring equinox than it does near the summer solstice, is a matter of mere spherical geometry, and has nothing to do with the kind of anomaly which Ptolemy is keen to explain. Even if the sun moved at a perfectly uniform rate around the ecliptic from our point of view, it would still be true that the daylight would increase more quickly and noticeably around the equinox than near the summer solstice. Suppose the earth is at E, and H is the horizon of some observer (you), and Q is the celestial equator, and the sun makes its daily circle first at A, then at the equinox at Q, then at B, then it climbs north and makes a circle at J, and then finally at the summer solstice K. (We are looking at all these circles of the sun's motion edge-on.) If we let arc AQ equal arc BQ, and again arc AB equal arc KJ, then it will take the same amount of time for the sun to climb from A to B and again from J to K. But AB is at right angles to EQ, so if we complete rectangle ABCD, then GC is the amount of daylight (or rather, half the amount of daylight) gained during the time the sun goes from A to B. But JK is tilted. So if we draw ML parallel to it, clearly LP (or rather, double LP) is the amount of daylight gained during the time the sun goes from J to K. And plainly, since PM = GD, and these are in line with each other, it follows that PL < GC. Hence there is less gain of daylight during the time from A to B near the equator than there is during the equal time from J to K near the solstice. And that will be true even if we suppose that the sun moves at a perfectly uniform rate around the ecliptic.

But Ptolemy sees that if we plot where the sun is on the ecliptic throughout the year, and note the dates, it spends unequal amounts of times in sweeping out equal angles around us. It does not move at a uniform rate, but speeds up, then slows down, then speeds up again.

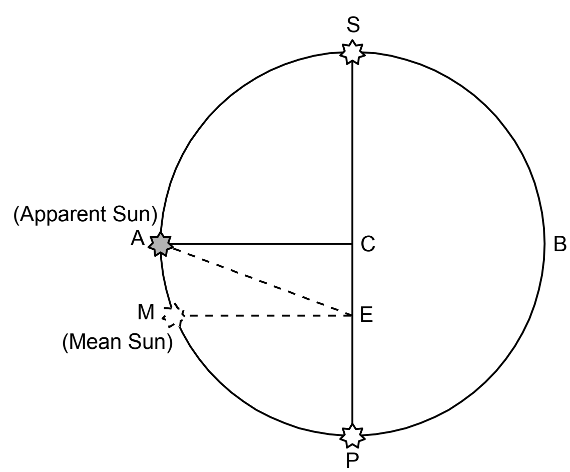
THE "MEAN SUN"

If the sun appears to us to move faster or slower along the ecliptic, then it is sometimes ahead of, sometimes behind, where the sun would appear to be if it moved uniformly around us along the ecliptic with the same period. Since Ptolemy accepts the "Astronomer's Axiom" that every celestial body's motion can be explained by uniform motion along perfect circles, he cannot simply take the apparent irregularity in the sun's speed at face value. He must explain it as the product of perfectly regular motion on a perfect circle (or on several perfect circles). We will see how to do this in more detail on another day. But for now, for the sake of explaining the concept of the "**mean sun**," let's look at one way this might be done. Suppose the circular path on which the sun makes its annual eastward motion around us does not have its center on us at E, but instead its center is at C. Since its center is off the center of the universe (us), therefore it is called an "off-center" circle, or an "**eccentric circle**." If we assume that the sun is moving with uniform speed on such a circle, sweeping out equal angles in equal times around C, then it will take the same time to go from S to A as to go from A to P, since ∠SCA = ∠ACP = 90°. But what will it look like its doing to us at E? It will appear to go first through ∠SEA (which is less than 90°), and then, in the same amount of time, it will go through ∠AEP (which is greater than 90°). So the sun will appear to speed up from our point of view, even though it is really moving uniformly on its own perfectly circular orbit.

Again, if M is where the sun would appear to have gone through 90° from S from our point of view, then although ∠SEM = ∠MEP, the sun will actually take longer to go through ∠SEM than ∠MEP, since ∠SCM > ∠MCP (and these are the angles that really matter, since C, not E, is the center of the uniform rotation). From our point of view, then, the sun would be going faster through some quadrants than others. But since SCP is a straight line, the time from S to P (via A) will be the same as the time from P back up to S (via B), since each of these arcs is 180° on the actual orbit of the sun. But in those places, the sun will also have appeared to go through 180° around us (unlike, say, if it goes from B to A, which is really 180° on its own circle, but appears less than that to us). That means if we can two spots in the ecliptic which are 180° apart from our point of view, and between which the sun spends exactly half its period, we will have found the points S and P in the zodiac, and so the orientation of the line SCEP in the stars will be known to us.

Suppose we have done that. Then position S is called the sun's "**apogee**," since that is where it is furthest from earth, and position P is called the sun's "**perigee**," since that is where it is closest to earth. And the straight line through those points and the center of the orbit and our eye, SCEP, is called the solar orbit's "**line of apsides**," since it passes through the "apses," that is, the highest and lowest points in the orbit.

Since we know the sun's period (as determined above), we know exactly how much time it takes to go any number of degrees on its own orbit, and we know exactly how many degrees it has gone on its own orbit in any given amount of time. So, after observing the sun at S, the apogee, suppose later in the year the sun is at A. To us, it looks as though it has only gone ∠SEA from apogee in that time. But



since we know it is orbiting us on an eccentric circle, and since we know how far it must have really gone on its own orbit in that time, we look at our table of the sun's regular movement (discussed above), and realize that the sun has actually gone through ∠SCA during the given time from apogee. If we now draw ∠SEM equal to ∠SCA, and imagine the sun at M, we will have an image of where the sun *would appear to us if it were moving uniformly around us instead of around C*. This imaginary or fictional sun, which moves uniformly around us just as the real sun moves uniformly around C, is called the "**mean sun**." By contrast, the location of the visible, physical sun, as it appears to us at A, is called the "**apparent sun**." Obviously, the mean sun always lies at the end of a line drawn from E parallel to CA. These two lines coincide at S and at P, but the mean sun is ahead of the apparent sun from S down to P, and the opposite is true from P back up to S. And the angle between the apparent sun, us (on earth), and the mean sun, or ∠AEM, is the angle of the sun's anomaly at any given time.

There is a certain ambiguity in the term "Mean Sun." On the one hand, it signifies a fictional projection of the sun's mean movement, and so is opposed to the "true" or "physical" sun, which is not really where we project the mean sun, but is where it appears to be, and coincides with the apparent sun. On the other hand, the mean movement of the sun means its regular (and hence "true") movement around its own center, as opposed to the apparent speed it seems to have around us. So in one sense "mean" is opposed to "true," but in another sense "mean" is opposed to "apparent."

We will get into more detail about these things soon.

PTOLEMY

DAY 17

TWO BASIC HYPOTHESES

We have seen that Ptolemy not only thinks that the nature of the heavens must be simple and uniform in some way, despite any apparent irregularities in their movements, but he has also specified the form which he thinks the simplicity and uniformity must take. He postulates that the heavenly bodies produce all their apparent motions by uniform circular motions. That is, no matter how irregular their movements might seem, really those appearances are produced by the stars moving at perfectly uniform speeds around perfect circles.

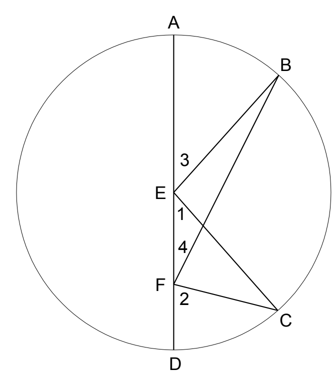
Later, we will see that the facts force him to loosen the rigor with which one might expect him to apply this principle. But for now, let's see what his basic proposals will be in order to produce the appearances of the planets by means of perfectly circular motions.

He introduces two "primary simple hypotheses." So he will introduce secondary ones, perhaps. Also, these are "simple," implying that he might compound them later. These two elementary models which will at once make the stars move in perfectly uniform circular motions, but also produce their apparent anomalies, are the following.

(1) FIRST SIMPLE HYPOTHESIS:

THE ECCENTRIC CIRCLE. (Or the "off-center circle")

The first hypothesis to explain apparent irregularity, or speeding up and slowing down, is the *eccentric* (*"off-center"*) *circle*. We have already seen this explained generally back in Day 16. Here, we will prove a little property of this hypothesis:



Suppose a circle with center E, diameter AD, your eye at F.
Suppose arcAB = arcCD.
Then TimeAB = TimeCD for a heavenly body.

Prove: $\angle 2 > \angle 4$

	$\angle 2 > \angle 1$	(it is external to $\triangle$EFC)
so	$\angle 2 > \angle 3$	($\angle 1 = \angle 3$, since arcCD = arcAB)
and	$\underline{\angle 3 > \angle 4}$	(it is external to $\triangle$EBF)
so	$\angle 2 > \angle 4$	

Which means that the angle through which the body appears (to you) to move in the time from C to D is greater than the angle through which the body appears to move in the time from A to B. But those times are equal. So the body appears to move through different angles in the same time, and so to speed up and slow down.

If F is where the eye is, i.e. the earth, then the body appears to move more quickly around PERIGEE (the point on eccentric circle that is nearest the earth), and more slowly around APOGEE (the point furthest from the earth). And EF is called the ECCENTRICITY, the amount of off-centeredness.

One image I like to use is that of a racetrack. If you are sitting inside the track, significantly away from its center, a car that is moving around the track at a constant pace of 100 mph will appear to move very quickly when near you and much more slowly when far away.

A QUESTION ABOUT THE ECCENTRIC.

Does the proof work for any point along the diameter besides the center? Can we say "From any other point than the center, the uniform circular motion will appear non-uniform"? What if we put point F, our eye, at D, the endpoint of the diameter? Then the star will come crashing into us, perhaps, but as long as it moves at a uniform rate through equal arcs on the circle, it will actually appear to move at a uniform rate from point D as well (or any point on the circumference), not just from point E! Why? Because the angles from F standing on equal arcs of the circumference must be equal—an elementary theorem of the geometry of circles.

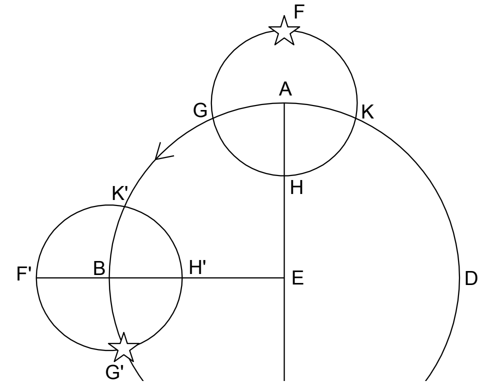
(2) SECOND SIMPLE HYPOTHESIS:

THE EPICYCLE (Or the circle "upon a circle").

This is the other elementary device for explaining apparent irregularity in the heavenly movements. The DEFERENT is the circle "carrying around" the epicycle, its center being E, where the eye or the earth is located. The EPICYCLE is the circle whose center lies on the deferent, and which is being carried around the deferent. The motion of the star around the center of the epicycle is uniform, and likewise the motion of the epicycle's center around the center of the deferent is uniform. But these two regular motions do not have to have the same speed as each other, nor do they have to be in the same direction. Those are adjustable options.

Suppose the star is at F when the epicycle is at A.
Later, the star is at G' when the epicycle is at B.

Then the regular motion (of the epicycle around the deferent) = ∠AEB.
The apparent motion (of the star) = ∠FEG'.



Hence the apparent motion exceeds the regular by arc BG', which is equal to arc AG.

WITH THE ECCENTRIC, THE SLOWEST SPEED IS ALWAYS AT APOGEE.

This is obvious. And fastest motion is always at perigee. But what about with the epicyclic hypothesis? There we have more options ...

THE SAME-DIRECTION EPICYCLE.

In the epicyclic hypothesis, there are TWO real motions going on, unlike in the case of the eccentric. We have (1) the motion of the star on the epicycle, and (2) the motion of the epicycle on the deferent. These do not have to be the same speed, nor do they have to be in the same direction. Each is uniform, but they need not be the same uniform speed or direction.

So we can speak of a SAME-DIRECTION EPICYCLE or an OPPOSITE-DIRECTION EPICYCLE (that is what I call them, anyway).

Ptolemy says that it is a property of the same-direction epicycle that the star appears to move *fastest at apogee*, e.g. at F in the diagram. That is because the motion of the star and of the epicycle cooperate to produce longitudinal progression. But when the star is at H on the epicycle (i.e. when it is closest to us, hence at perigee), there the two motions will conflict the most, and so the star's motion on the epicycle will do the most to "undo" the apparent progression of the star in longitude. So, unlike the hypothesis of the eccentric, it is possible for the star on a same-direction epicycle to appear fastest when it is furthest from us, slowest when it is closest to us.

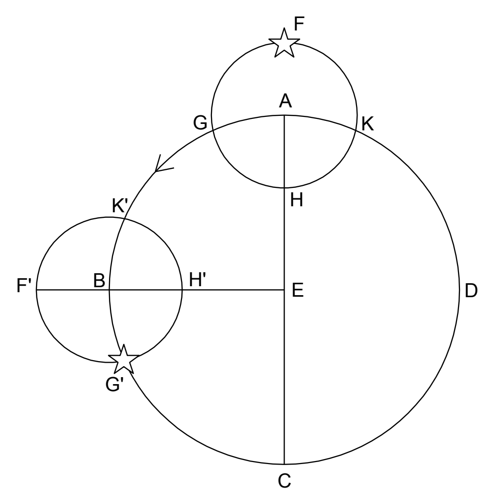
Note that since the eccentric & the same-direction epicycle have opposite properties, they cannot be exactly equivalent to each other.

Also (here is a bit of a spoiler) Ptolemy will use same-direction epicycles to explain the planetary motions.

THE OPPOSITE-DIRECTION EPICYCLE.

On the other hand, the opposite-direction epicycle is clearly like the eccentric. If the star is at H and moving clockwise around A while the epicycle is moving counter-clockwise around E, plainly the star will appear fastest to us there, at H. There it is not only closest to us, but also the two motions are most cooperative. Conversely, when the star is at F, it is not only furthest from us, but the 2 motions are also most in conflict. Hence the star appears slowest at apogee.

Ptolemy uses this hypothesis as an equivalent for an eccentric, e.g. for the sun's annual motion.



A QUESTION ABOUT THE EPICYCLE: Suppose we draw a line from the center of the deferent to where the epicycle cuts the deferent, as DS. Will DS be tangent to the epicycle? No! DS often looks tangent to the epicycle, but in fact it cannot be tangent to it.

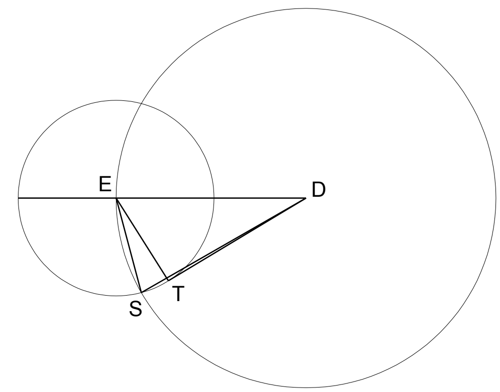
If it were, ∠ESD would be a right angle.

But ED = DS (being radii of the deferent), so that △EDS is isosceles.

Hence ∠SED would also be a right angle, which is impossible.

Therefore, if we now draw DT tangent, T is not on the deferent. More specifically, since ∠DTE would be a right angle, and DE the hypotenuse, it follows that DE > DT, and therefore T lies inside the circle of the deferent.

On the other hand, if we are not at D, but at some other point inside the circle, i.e. if the deferent is itself an eccentric circle, it might then happen that the line from our eye to S is tangent to the epicycle. So don't think it simply can never be done! But the line from D to S can never be tangent to the epicycle.

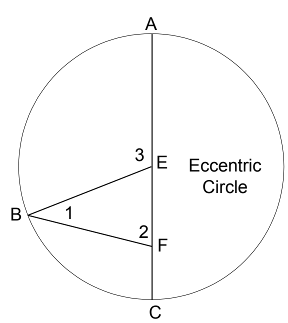


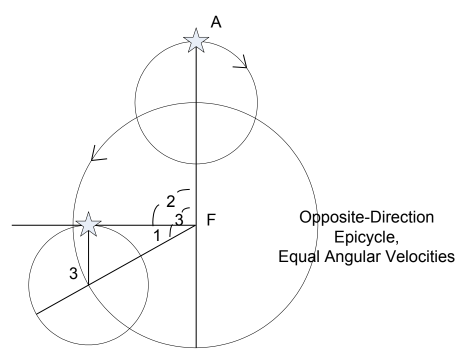
COMBINING THE HYPOTHESES, and EQUIVALENCE.

Ptolemy has hinted already that he will be combining the hypotheses, i.e. he will have epicycles moving on eccentric deferents. You could even have an epicycle on an epicycle, if you want!

He also says, in Chapter 3 of Book 3 of the *Almagest*, that it is possible for two distinct hypotheses to be equivalent, and gives conditions under which certain hypotheses are equivalent—but we will discuss equivalence in Day 18.

WHAT IS AN "ANOMALISTIC DIFFERENCE"?
Ptolemy speaks of "**anomalistic differences**." What does he mean by that?

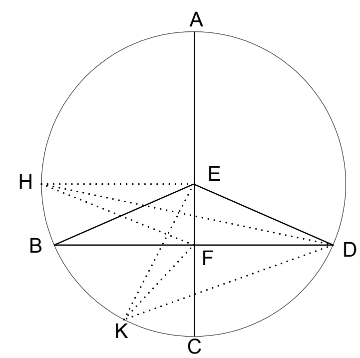




It is the angular difference between a star's true and regular movement (angle 3) and its apparent and irregular movement (angle 2). That difference is angle 1 [by Euclid 1.32 in the eccentric, and by simple subtraction in the opposite-direction epicycle].

Ptolemy also uses the terms "**greatest passage**" and "**mean passage**" and "**least passage**." What Ptolemy calls the point of "greatest passage" we would today call the point in its orbit at which a planet (or the sun or the moon) has the greatest apparent speed. Similarly, what he calls the point of "least passage" would be the point at which its apparent speed is slowest. And what he calls the point of "mean passage" we would call the place where it appears to move with its average apparent speed around us, or, what comes to the same thing, where it appears to move with its mean speed, i.e. with its actual uniform speed. But since these are exactly points in its orbit, not whole chunks of its orbit, we cannot talk this way without introducing the idea of an instantaneous speed, which is not something that Ptolemy would have done. So instead he makes these points *dividing points*. The point of "greatest passage" is the point dividing the planet's motion such that up to that point the planet appeared to be speeding up, but after that point it appears to be slowing down. And the point of "least passage" is the point dividing the planet's motion such that up to that point the planet appeared to be slowing down, but after that point it appears to be speeding up. And the point of "mean passage" is one of two points: one divides the planet's motion such that up to that point it appeared to be moving faster than the mean speed, but afterward it appeared to be moving more slowly than the mean speed, while the other one is the reverse—prior to it, the planet's apparent motion was slower, afterward faster, than the mean speed.

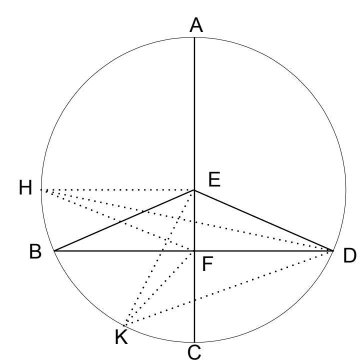
GREATEST ANOMALISTIC DIFFERENCE THEOREM.



Given: Eccentric with center E,
Earth at F,
Apogee at A,
Perigee at C,
Apparent 90° from apogee at B,
i.e. $\angle AFB = 90°$.

Prove:
- $\angle EBF$ is the greatest anomalistic difference
- B is the point of mean passage
- Time (from least passage to mean) > Time (from mean to greatest passage)

Take random points H and K on the circle on either side of B.



I say that	$\angle EBF > \angle EHF$	
and	$\angle EBF > \angle EKF$	
For	$HF > BF$	[Euclid 3.7; 3.3]
so	$HF > FD$	[BF = FD]
so	$\angle HDF > \angle DHF$	

but	$\underline{\angle EDH = \angle EHD}$	[Euc. 1.5]
so	$\angle EDF > \angle EHF$	[sums]
i.e.	$\angle EBF > \angle EHF$	[since $\angle EDF = \angle EBF$]

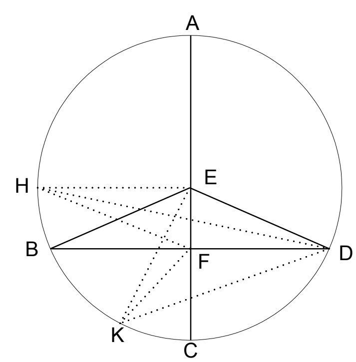
Again	$DF > KF$	[Euc. 3.7; 3.3]
so	$\angle FKD > \angle FDK$	
but	$\angle EKD = \angle EDK$	[Euc. 1.5]
so	$\angle EKF < \angle EDF$	[remainders]
i.e.	$\angle EKF < \angle EBF$	[since $\angle EDF = \angle EBF$]

Hence $\angle EBF$ is the greatest anomalistic difference.

Q.E.D.

I say next that B is the point of mean passage, i.e. the point such that when the sun is there, its apparent speed is the same as its regular speed, or the point dividing arc ABC such that in arc AB the sun appears to be moving slower than its mean speed, but in arc BC it appears to be moving faster than its mean speed.

Think of the apparent movement and the regular movement as being analogous to two cars on a highway. As long as apparent is slower than regular, the difference in the distances they cover in more and more time gets greater and greater. That distance between them *stops* increasing precisely at the moment when the apparent *begins* to catch up with the regular, i.e. when apparent *stops* being slower and *starts* being faster than the regular.



But when does the apparent stop being slower and start being faster than the regular? *As soon as they are going the same speed.* Hence at the very same instant they are going the same speed they have also reached the maximum difference between them.

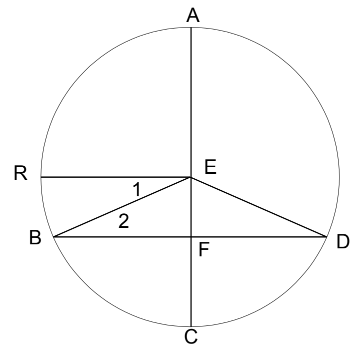
Therefore the point of greatest anomalistic difference, namely B, is the point of mean passage. But the star is apparently slower than the true (mean) speed all the way from A to B, and faster all the way from B to C.

And since	arcAB > arcBC,
thus	Time(thru arc AB) > Time (thru arc BC)
so	Time[least to mean] > Time[mean to greatest]

Q.E.D.

COROLLARY.

As a corollary, we now also see that "arc AB > arc BC by twice the arc containing the greatest anomalistic difference."



For if we draw ER parallel to FB,

then $\angle 1 = \angle 2$

so arcRB = arc of $\angle 2$ (the arc which $\angle 2$ would stand on if its vertex were placed at E)

i.e. arcRB = arc of greatest anomalistic difference

but arcAB = arcAR + arcRB

so arcAB = arcRC + arcRB [arcAR = arcRC]

i.e. arcAB = [arcBC + arcRB] + arcRB

so arcAB = arcBC + 2arcRB.

i.e. arcAB = arcBC + twice the arc of the greatest anomalistic difference.

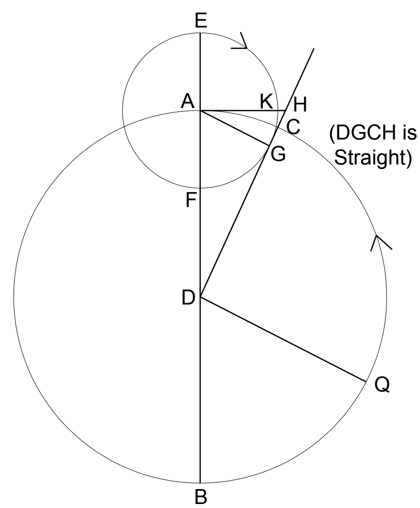
Q.E.D.

PTOLEMY

DAY 18

GREATEST ANOMALISTIC DIFFERENCE AND MEAN PASSAGE IN THE EPICYCLIC HYPOTHESIS;
FIRST EQUIVALENCE PROOF

In Day 17, we were introduced to the two primary simple hypotheses which Ptolemy will employ in order to explain the anomalies in celestial movements by means of perfectly uniform circular motions. These were the eccentric and the epicyclic hypotheses. We also learned some of their basic properties. But while we learned where the greatest anomalistic difference occurs in the case of the eccentric, and also where least, greatest, and mean passage occur, and how the times between these are unequal, we have yet to learn these things in the case of the epicyclic hypothesis. So let's do that next. We'll start with the opposite-direction epicycle, which is more like the eccentric hypothesis than the same-direction epicycle is. We will also consider a simple case, where the speed of the star on the epicycle is the same as the speed of the epicycle on the deferent.



To find the point of the greatest anomalistic difference for the EPICYCLIC hypothesis:

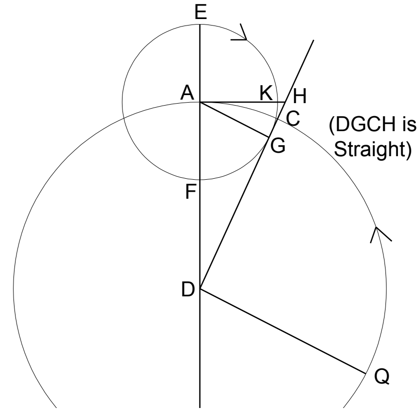
Given: Opposite-direction epicycle A on deferent D
Apogee occurs out along line DQ
Star speed = epicycle speed
Star is appearing at G, one quadrant from apogee
[i.e. ∠QDG = 90°]

Prove: The anomalistic difference when the star is at G is the greatest.

The star has gone from E to G, and the epicycle from Q to A.

So ∠QDA = ∠EAG
so AG is parallel to DQ
so ∠AGD = ∠GDQ
so ∠AGD = 90°
so DG is tangent to the epicycle
so star at G is appearing as far from A as it can, i.e. it is furthest off from the mean motion that it can get.
So ∠ADG is the *greatest* anomalistic difference.

Q.E.D.



NOTE: There is only one other point of tangency on the epicycle and so the greatest anomalistic difference occurs twice.

And G is also the point of MEAN PASSAGE, because at G the star on the epicycle is coming straight at the observer D, and thus contributes nothing to the apparent speed, so the apparent speed is only the motion on the deferent, i.e. the mean speed.

Also, unlike the case of the eccentric, in this epicyclic hypothesis we can separate the cause of the anomaly, namely the motion on the epicycle. And the "MEAN MOTION" is the same as the AVERAGE SPEED of the star around us, which will also be the same as the speed of the epicycle around the deferent.

THE REST OF THE PROOF.

We also can say, as we did in the case of the eccentric, that

Arc EG > arc GF,
hence the Time [Least to Mean] > Time [Mean to Greatest]
i.e. Time from apogee to G > Time from G to perigee

Also: arcEG – arcGF = 2arcAC
where arcAC is that of the greatest anomalistic difference, or ∠ADG.

For:

arcEG – arcGF = [arcEK + arcGK] – [arcFK – arcGK]
so arcEG – arcGF = arcEK + arcGK – arcEK + arcGK
so arcEG – arcGF = 2arcGK
so arcEG – arcGF = 2∠KAG
so arcEG – arcGF = 2∠ADG (since △ADG similar to △AGH)
so arcEG – arcGF = 2arcAC (in *degrees*, not in actual length)

And we can measure arcAC if we plot the course of the mean planet.

WHAT ABOUT THE SAME-DIRECTION EPICYCLE?

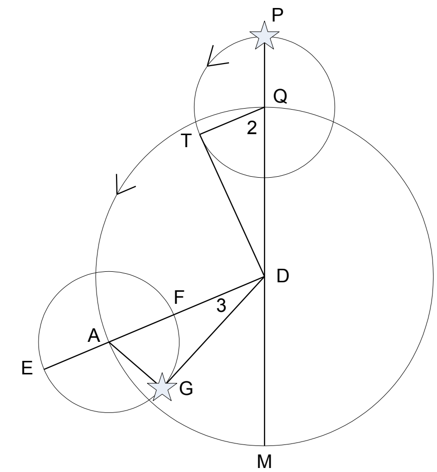
In Chapter 3 of Book 3 of the *Almagest*, Ptolemy says if we use a same-direction epicycle instead, we get the opposite result from the above, i.e.
Time [Greatest to Mean] > Time [Mean to Least]

He uses this fact in his planetary theory later in order to determine the sort of epicycle at work, so it is not a small point.

Take such an epicycle, with star initially at apogee P.
Draw DT tangent to the epicycle at Q.
Draw DA parallel to QT.
Draw the epicycle at A.

Since the epicycle went through ∠QDA, therefore the star is at G where ∠EAG= ∠QDA.

But	∠PQT = ∠QDA	[DA is parallel to QT]
so	∠EAG = ∠PQT	
so	DG is tangent to the epicycle at G.	
Hence	G is the point of mean passage.	
So	Time of epicycle thru arcQA is Time[Greatest to Mean]	
and	Time of epicycle thru arcAM is Time[Mean to Least]	
and	arcQA > arcAM (since arc QA is similar to arcPT > 90°)	



Q.E.D.

AN OBJECTION TO WHAT PTOLEMY SAYS ABOUT THE SAME-DIRECTION EPICYCLE.

Ptolemy says that in the same-direction epicycle "there will result at the apogee the greatest advance, because the epicycle and the star are moving the same way." He also says that

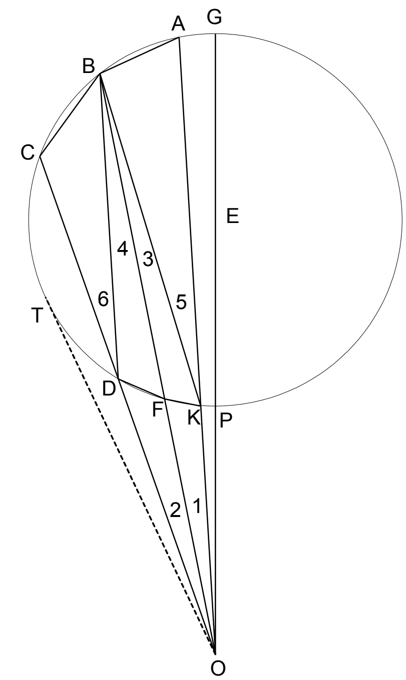
Time [from greatest to mean passage] > Time [from mean to least passage]

and he says this follows because "in this case the greatest progress is effected at the apogee."

Later, in the planetary theory (in Book 9 Chapter 5 of the *Almagest*, to be precise), Ptolemy will decide that Venus must be on a SAME-direction epicycle because, for Venus, Time[greatest to mean passage] > Time[mean to least passage].

So what he is saying now, in Chapter 3 of Book 3, ends up being very important later, and it is not just possible, but NECESSARY that if we have a same-direction epicycle then the fastest apparent speed must be at apogee. But *we can object to what he says that idea in this way: Why must we say that the fastest motion occurs at apogee? Yes, the two motions cooperate most there, but then again the star is farthest away there, and motions farther away seem slowe*r. So what stops us from supposing that if the epicycle is LARGE enough, then the "farness" at apogee outweighs the "cooperativeness" of the two motions there, so that the time of greatest apparent movement is actually somewhere *after* apogee, when the star is closer to us?

But it turns out Ptolemy is right. Even though the star is farther away at apogee, and that fact will tend to make its truly-regular motion appear slower to us, nonetheless that is outweighed by the cooperativeness of the two motions at that place.



To see this, consider the following proof.

Given: Epicycle E, observer O
Star moving from apogee G in any 2 equal consecutive arcs AB, BC prior to tangency at T.
∠1 = AOB
∠2 = BOC

Prove: ∠1 > ∠2

Prior to tangency, the movement of the star and of the epicycle still cooperate (although not as much as at G), but the star is also getting closer to us. After tangency, the motion of the star is counteracting that of the epicycle, so it will not then appear to move faster (in the counter-clockwise direction around O). So if the star can seem to move faster than at apogee, it must be while it is along arc GABCT somewhere. And since the star's apparent speed smoothly increases to its maximum and slows down again to its minimum, if the star is not at its maximum apparent speed at G, it must appear to speed up after that, so that somewhere along arc GBT, in two equal consecutive times, i.e. in the times it takes to go two equal consecutive arcs along the epicycle, like AB and BC, it will appear to go faster in the second than in the first. In other words, ***if*** the star does not appear to move fastest at G, ***then*** there should be two equal arcs like AB and BC in which it appears to move faster through the second than through the first. Hence the apparent ∠1 should be less than the apparent ∠2. And so if that is never true, then neither is it true that the fastest apparent motion is anywhere but at the apogee.

Now join AB, BC, DF, FK, KB, BD.

∠CBF is supplement to ∠CDF [Euc. 3.22]
i.e. <u>∠FDO is supplement to ∠CDF</u>
so ∠CBF = ∠FDO
i.e. ∠CBO = ∠FDO

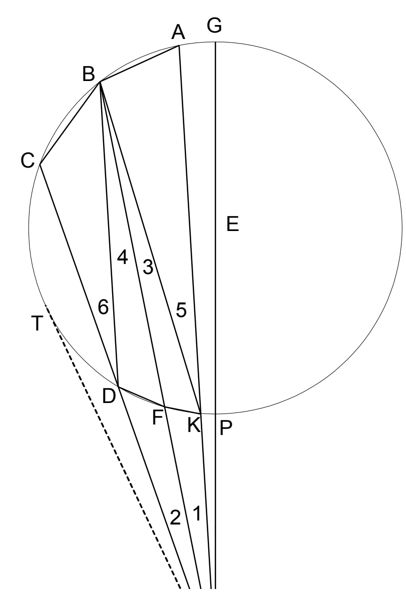
But ∠2 is in △CBO and in △FDO.

Hence △CBO is similar to △FDO.

Likewise △ABO is similar to △FKO.

And so BC : BO = DF : DO
and BA : BO = FK : KO

But BC = BA

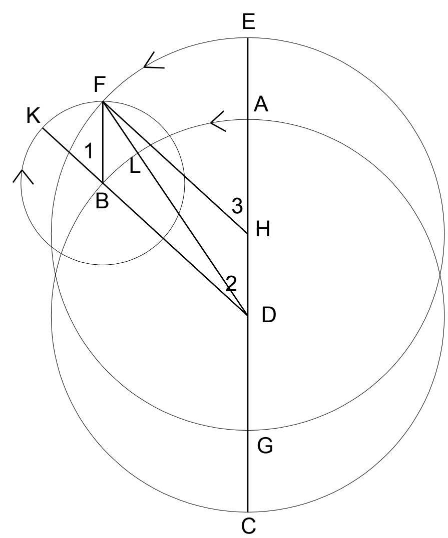


so	DF : DO = FK : KO	
But	DO > KO	[Euc. 3.8]
so	DF > FK	
so	arcDF > arcFK	
so	$\angle 4 > \angle 3$	
Now	$\angle 5 = \angle 6$	[since arcAB = arcBC]
But	5 = 1 + 3	[Euc. 1.32]
and	6 = 2 + 4	[Euc. 1.32]
so	1 + 3 = 2 + 4	
But	$\angle 4 > \angle 3$	
so	$\angle 1 > \angle 2$	

Therefore there can be no place after apogee where the star appears to move more quickly.

Q.E.D.

THE FIRST EQUIVALENCE PROOF.



After introducing his two primary simple hypotheses, Ptolemy shows that it is possible for them to be equivalent to each other, that is, it is possible for the eccentric hypothesis and the epicyclic one to produce exactly the same appearances to us, if we adjust them the right way. For instance, if we are at D, then a star moving through $\angle 3$ on an eccentric circle of center H, with radius HE, will produce certain appearances, for instance it will appear fastest at G, slowest at E. But the exact same appearances will result, throughout the motion, if we assume instead that the star is moving on an opposite-direction epicycle with radius BF = DH, and if the epicycle speed is the same as that of the speed of the star on the eccentric, and if the star on the epicycle travels with the same angular velocity (although in the opposite direction) as the epicycle on the deferent.

Given:
- Eye at D
- Opposite-direction epicycle with radius BF
- Concentric deferent AC
- Eccentric with diameter EG = AC, center H, DH = BF
- All 3 angular speeds equal
- Epicycle has gone arcAB (in some random time) from apogee
- F is *upper* point of intersection of eccentric & epicycle

Prove: Star on eccentric is at F
Star on epicycle is also at F

	DB = FH	[given]
	DH = BF	[given]
so	HDBF is a parallelogram	
so	1 = 2 = 3	
so	arcKF is similar to arcAB is similar to arcEF	
so	star on epicycle is at F, and star on eccentric is at F.	

Q.E.D.

Note that

∠HFD is the eccentric anomalistic difference
∠BDF is the epicyclic anomalistic difference

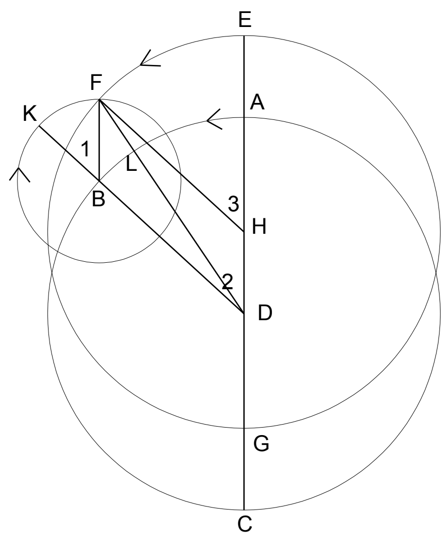
Imagine the circle AC, or even draw it on a chalkboard. Now take a piece of chalk equal to DH or BF. Holding this chalk always parallel to DH or BF, and tracing out the circle AC with the *bottom* end of the chalk, the *top* end will trace out the circle EG, which is equal to circle AC, just transferring it up by the length of the chalk, DH. In just this way, the radius of the epicycle is always parallel to itself, and the star at the end of it must always lie on the eccentric. Beautiful!

IMPLICATIONS OF EQUIVALENCE.

What are the philosophical implications of this equivalence of these two hypotheses or models?

To the extent that they are completely equivalent, we are unable to determine which one is really going on (if either of them).

These things are mathematically equivalent, in terms of “where the body will be when,” but they are not physically equivalent. That is, if one of them were really happening, we are positing a very different piece of machinery from what the other one would require us to suppose exists in the heavens.



PTOLEMY

DAY 19

MODEL TO ACCOUNT FOR THE SOLAR ANOMALY

We have now followed Ptolemy through quite a long series of preliminaries. We have learned his model for the universe in a general way without quantitative specificity. We have built up a table of chords and arcs for trigonometric purposes. We have also found ways to determine arcs and chords on a sphere given certain other chords or arcs. And we have learned the properties of the two primary simple hypotheses he intends to use in order to explain the anomalies in the movements of the sun, moon, planets and stars. We are now ready to begin applying what we have learned in order to develop quantitatively specific models for the movements of the heavens.

We will follow Ptolemy and begin with the sun. One reason to do this is that the other anomalies of the planets (as we shall see) seem to be somehow tied to the sun, and their patterns are determined by it. In Chapter 4 of Book 3 of the *Almagest*, Ptolemy also explains that we are starting with the solar anomaly, as opposed to some planetary one, "because there is only one," whereas for a planet there is a zodiacal and also a heliacal anomaly (to be explained later).

What is the solar anomaly? Ptolemy is so quick to specify the nature of the anomaly, that he almost skips past telling us what it is, or how we observe it. But the numbers from Hipparchus make it clear:

Days from Spring Equinox to Summer Tropic = 94.5
Days from Summer Tropic to Autumn Equinox = 92.5

(Ptolemy confirms these numbers by his own observations.)

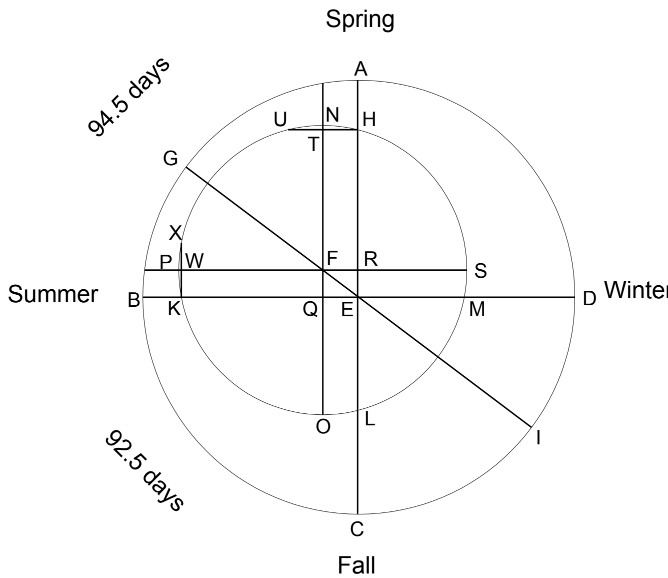
But the angular movement from spring equinox to summer tropic is 90°, and the angular movement from summer tropic to autumn equinox is also 90°. So the sun moves faster in the 90° from summer tropic to autumn equinox than it does in the 90° from spring equinox to summer tropic.

So either the sun is not moving uniformly on its circle, or else we are not at the center of its regular motion (eccentric), or it is not moving on just one circle (epicycle). But we can't say that it is not moving uniformly on its own circle, since that will destroy the intelligibility of astronomical phenomena (as Ptolemy conceives of it).

So we will explain this anomaly first by means of an eccentric circle. That leaves us with two questions (paragraph 2):

(1) What is the ratio of eccentricity for the ecliptic?

(2) Where on the ecliptic does apogee occur? (In other words, what is the orientation of the line of apsides, the line joining the earth and the center of the eccentric circle?).



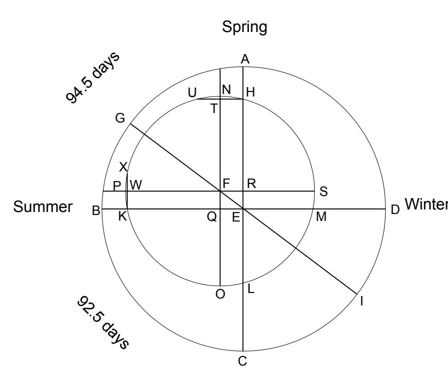
FINDING THE ECCENTRICITY OF THE SUN'S YEARLY MOTION.

(Note: since this diagram looks like a fried egg, students sometimes refer to this as the "Fried Egg Prop." Giving names like that to propositions is not just fun, but it can also make them more memorable.)

Given: Spring Equinox to Summer Tropic = 94.5 days
Summer Tropic to Fall Equinox = 92.5 days

Find: The ratio of the eccentricity to the radius of the eccentric
(i.e. the ratio of EF : FO)

Let circle HKLM, center F, be the Sun's eccentric circular path.
Let E be the earth (our eye), center of the ecliptic (circle ABCD), which is the projection of the sun's actual path onto the celestial sphere.



A = Spring Equinox
B = Summer Tropic
C = Fall Equinox
D = Winter Tropic
Now the sum of days from spring equinox to fall equinox is 187 days, which is more than 365 ÷ 2 (i.e. 182.5), and so this course is more than half the year. So arc HKL on the sun's

path must be more than 180° (because in its regular movement, in more than half the time, it must complete more than half its circle), i.e., we know we have placed the center F correctly in semicircle ABC since the sun spends more time on arc ABC than it does on arc CDA. Hence the center of its motion lies to the left of AEC.

Again, since the sun spends more time in arc AB than in arc BC, the center of the sun's eccentric path is above BED. So it lies within quadrant AEB.

Draw NFO parallel to AC, and PFS parallel to BD.

Draw HT at right angles to FN, and produce to U (hence HT = TU).

Draw KW at right angles to FP, and produce to X (hence KW = WX).

Now, since the sun goes from A to C (i.e. from H to L) in (94.5 + 92.5) days,

hence	arcHKL = (93° 9' + 91° 11')	[p.86, Table of Sun's regular motion]
so	arcHKL = 184° 20'	
so	arcHKL – arcNPO = 184° 20' – 180° = 4° 20'	
so	arcNH + arcOL = 4° 20'	
so	2arcNH = 4° 20'	
so	arcUH = 4° 20'	
so	UH = 4^P 32'	[Table of Chords, where NO = 120^P, i.e. 120 Parts]
so	HT = ½ UH = 2^P 16'	
so	EQ = 2^P 16'	
Again	arcHNPK = 93° 9'	[since sun goes H to K in 94.5 days]
so	arcHNPK – arcHN – arcNP = 93° 9' – 2° 10' – 90°	
so	arcPK = 0° 59'	
so	2arcPK = 1° 58'	
so	arcKX = 1° 58'	
so	KX = 2^P 4'	[Table of Chords, where NO = 120]
so	KW = ½ KX = 1^P 2'	
so	FQ = 1^P 2'	

So we have EQ and FQ. Therefore, by Euclid 1.47, we have

EF = 2^P 29' 30''

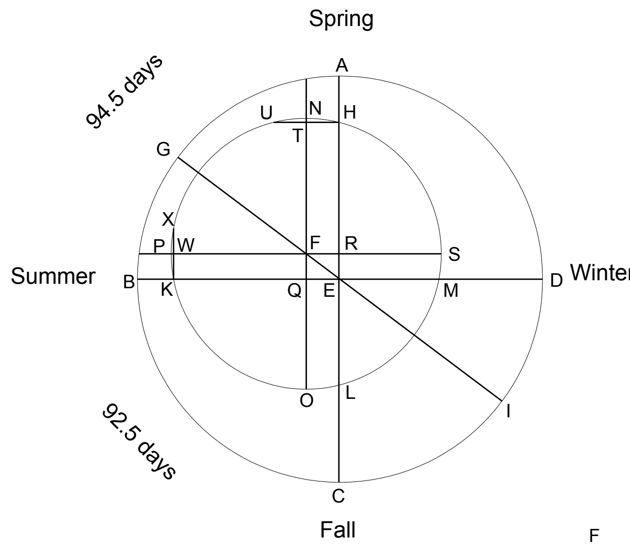
where the FO = 120, or the radius of the eccentric is 60 parts.
Hence (Distance from earth to sun's eccentric center) : (Distance from sun to sun's eccentric center) = (2 + 29/60 + 30/3600) : 60 = 2.5 : 60 = 1 : 24.

That is, EF : FO = 1 : 24

Q.E.I.

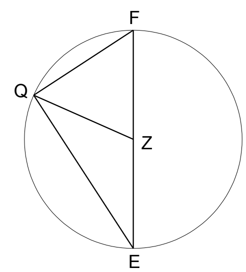
FINDING THE ORIENTATION OF THE SUN'S LINE OF APSIDES.

Looking back to the same givens and the same diagram, let's focus on that right triangle FQE. If we can find the value of ∠FEQ, i.e. of ∠GEB, then we will know the angular distance from the summer tropic to the sun's line of apsides.



Bisect the hypotenuse EF in right triangle FQE. Call the midpoint Z.

Now, by our results above, we know that



$$FQ : EF = (1^P\ 2') : (2^P\ 29'\ 30'') = X : 120$$

So FQ = 49^P 46' [when EF = 120]
so arcFQ = 49° [Table of Chords]
so ∠FZQ = 49°
so ∠FEQ = ½ ∠FZQ = 24° 30'
so ∠BEG = 24° 30'

Hence apogee (G) occurs 24° 30'–worth of ecliptic before B, the summer solstice, which turns out to be 5° 30' into the Twins (in the time of Ptolemy).

So, according to Ptolemy, the sun is furthest away from us between spring and summer. (And that is still considered true today.)

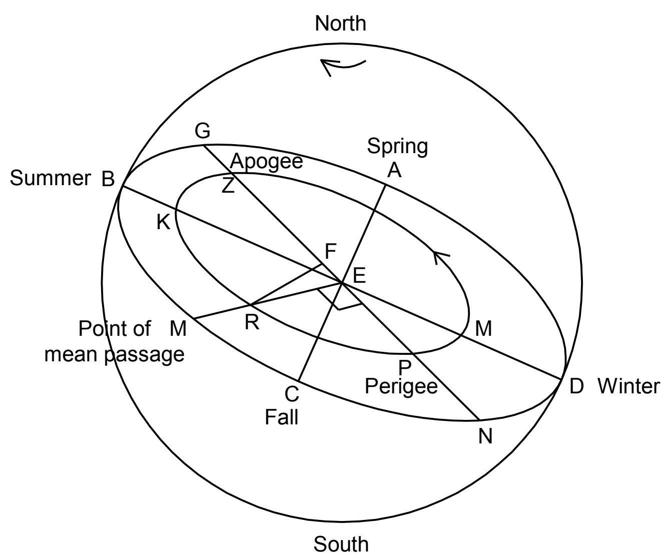
FINDING THE TIME IN THE LAST TWO QUADRANTS.

How long does the sun spend in arc LM?

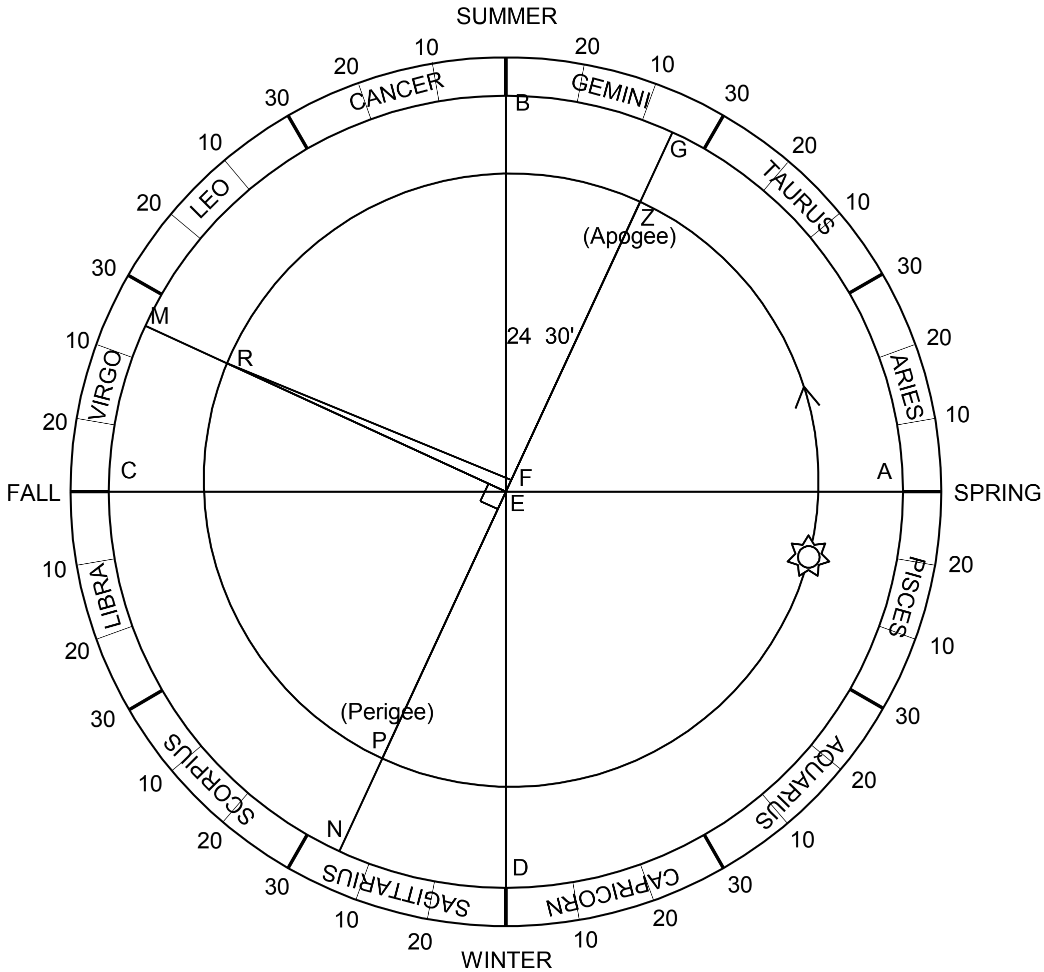
arcOS = 90°
arcOL = arcHN = 2° 10'
so arcMS = arcPK = 0° 59'
so arcOS – arcOL – arcMS = 86° 51'
i.e. arcLM = 86° 51'
so time in arcLM = 88 1/8 days [Table of Sun's Regular Movement]

How long does the Sun spend in arc MH?

and	arcSN = 90°
	arcHN = 2° 10’
	arcMS = 0° 59’
so	arcSN – arcHN + arcMS = 88° 49’
i.e.	arcMH = 88° 49’
so	timeMH = 90 1/8 days [Table of Sun’s Regular Movement]



Below is a diagram showing the Sun’s eccentric orbit around earth at E. F is the center of that circle, and the line of apsides, through EF, points to N and G in the zodiac. We saw that EF : FR = 1 : 24, and this drawing more or less reflects that. You can see that while the sun’s orbit is eccentric, it is not *very* eccentric!


SUMMER
WINTER
FALL
SPRING
CANCER
GEMINI
TAURUS
ARIES
PISCES
AQUARIUS
CAPRICORN
SAGITTARIUS
SCORPIUS
LIBRA
VIRGO
LEO
B
G
Z
(Apogee)
M
R
24 30'
C
F
E
A
(Perigee)
P
N
D

PTOLEMY

DAY 20

THE PRECESSION OF THE EQUINOXES

In the interests of brevity, we will skip now from the solar theory to the stellar and planetary theory of Ptolemy. In so doing, we pass over many things of interest in themselves, and things which were of great interest to Ptolemy and to ancient astronomers, such as the explanation and prediction of solar and lunar eclipses. But to keep this course finite, and to keep it focused on certain central themes with the most wide-reaching consequences (such as the shift from geocentrism to heliocentrism), we now leap ahead to Book 7 of the *Almagest* where Ptolemy considers the stars, beginning with the "fixed" stars (as opposed to the "wandering" ones, the planets).

In Chapter 1 of Book 7, Ptolemy explains that:

THE "FIXED STARS" ARE FIXED.

The "fixed" stars are fixed on (or in) their sphere (the celestial sphere) and relative to each other, but the sphere itself is not fixed. It does one full rotation per (sidereal) day.

(In reality, it is not exactly true that the stars are fixed relative to each other. Constellations very slowly change shape, and binary stars orbit a common center of gravity. The stars in our particular galaxy have many distinct motions, and some common ones, e.g. they drift with the galaxy away from all other galaxies, and they rotate about the common center of mass of the entire galaxy. But all of that is light years ahead of Ptolemy. I mention it here only because I don't wish to give the impression that the stars are truly fixed in their apparent relative positions—they just take so long to shift in their apparent relative positions that their motions were not discovered until relatively recently. And binary stars require telescopes, or at least binoculars, to be observed in their motions.)

Hipparchus thought (oddly) that only the Zodiac belt precessed, and not the other stars, as if some of the stars moved relative to the others. That was an error.

BOOK 7 CHAPTER 2: THE PRECESSION OF THE EQUINOXES.

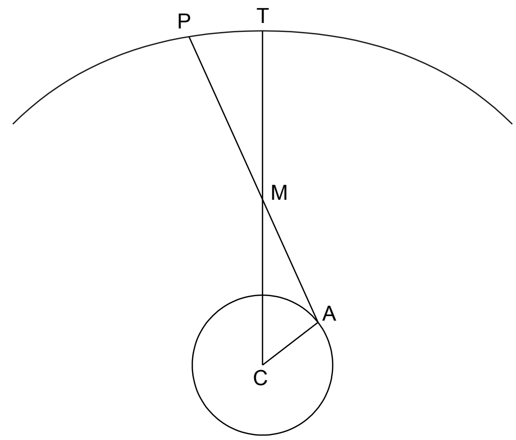
In this chapter of the *Almagest*, Ptolemy describes what sorts of observations confirm that the sphere of fixed stars not only has its daily rotation from east to west about the celestial poles, but also has a much slower rotation from west to east around the poles of the ecliptic. This phenomenon is known as "**the precession of the equinoxes**." Recall that the equinoxes are two distinct points on the ecliptic, namely where the sun's annual path (or its projection onto the sphere of fixed stars) intersects the celestial equator. As it turns out, those two points are not always the same points on the celestial sphere, and similarly the two solstice points are not always the same points on the celestial sphere. So there is some other motion going on. And we will have to decide whether to make it a motion of the sun's orbit, or else another motion of the celestial sphere itself. But first let's sketch out what sorts of observations confirm that there is some sort of motion of the equinoxes and solstices relative to the fixed stars. We will go into some detail here, in some ways more than is strictly necessary, in order to get a sense of what astronomers have to go through in order to ascertain certain motions in the heavens ...

In connection with the precession of the equinoxes, Ptolemy says "This research we have made by means of the instrument we prepared for the observation of the moon's particular elongations from the sun." The instrument to which he refers is an "astrolabe," a compact instrument (made of concentric graduated circles) for measuring the positions of stars, later made obsolete by the (now also obsolete) sextant. He described the construction of this instrument in Chapter 1 of Book 5 of the *Almagest*. But let's keep things simple and recognize that it would not be difficult to construct an instrument which one could mount on a tripod and orient diverse concentric circles on it along different lines of sight to different points in the heavens and thus take accurate readings of various apparent angles between those celestial points.

Also, in Chapter 2 of Book 7, he refers to "**lunar parallax**," and other points of lunar theory (which we have skipped over), in order to describe observations by which it can be determined that the Summer Tropic precesses. The moon is helpful in this regard, because although the sun's motion defines the location of the summer tropic, we cannot directly measure the angular distance between a fixed star and the sun (so as to determine the distance of the star from the summer tropic), for the very simple reason that we cannot generally or easily observe the sun and a fixed star at the same time. But we can observe the moon and a fixed star at the same time, and we can also observe the sun and the moon at the same time, so the moon becomes the connecting term. We can observe the sun and moon together just before sunset, for example, and then a half hour later observe the moon and the fixed star in question. (Some adjustments must be made for the half hour of difference between the observations, as we shall see.)

But back to "lunar parallax." This detail is not terribly important for understanding the precession of the equinoxes, but it is involved in Ptolemy's way of confirming the precession, and it is also worth knowing for itself. So let's take a moment to understand it in a general way. Ptolemy discusses how to observe the moon's parallaxes in Chapters 11 and 12 of Book 5 of the *Almagest* (from which he determines the ratio of the moon's distance from earth to the earth's radius in Chapter 13, although his value is not very good due to factors we need not enter into here). "Parallax" is the phenomenon by which the same object at the same instant appears against different spots on a more distant background depending on where one is viewing it from. If you hold your finger out at arm's length and look at it now with one eye, now with the other (closing one eye, then the other), you will see that your finger appears to shift back and forth on whatever background you view it against. If the background is much further away than your finger, say a mountain range, your finger will appear against very different places on that background—but if the background is not much further away than your finger, say if you hold your finger up to your computer monitor, almost touching it, your finger will then not appear against very different parts of the monitor from the two different points of view of your two eyes. And the distance between the two points of observation is important. If that distance is very small compared to the distance out to the object being observed, you might notice no parallax at all. For instance, if you look at a tree which is nearly a mile away, and observe where it appears against the mountain-range behind it using your left eye, and then again using your right eye (while standing in the same spot), you will not see much or any parallax. But if you observe that tree from one spot, then again from another spot 1 mile away along a line parallel to the mountain-range, you will certainly notice a parallax.

Because the earth's size is not significant enough compared to the distances out to the stars or even the planets in order to produce a sensible parallax for observers at different places on earth (telescopes are needed to get those parallaxes), Ptolemy could not get any parallax on those bodies. But because the earth has a significant size compared to the moon's orbit (the radius of the earth is in fact about a sixtieth of the mean distance to the moon), therefore an observer at A on earth will see the moon at P in the fixed stars, when its "true" geocentric position is out at T. This is a lunar parallax equal to ∠AMC.



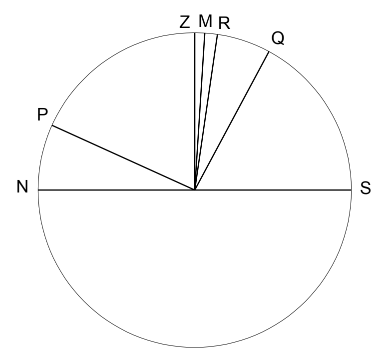
So how do we ever see a true position of the moon? If your latitude is southerly enough, the moon will sometimes pass through your zenith, and therefore there will be *no parallax* effect, because the line from the center of the earth to the moon will pass through you.

In Alexandria, where Ptolemy lived, Latitude = 30° 58'.

So arcZQ, the arc on his meridian between the zenith and the equator, = 30° 58'.

So arcZR, the arc on his meridian between the zenith and the summer tropic, = 30° 58' – 23° 51' = 7° 7'. (R is where the Summer Tropic cuts the Meridian.)

And the moon gets 5° north of that at its northernmost pass. So if we call the point where the moon cuts the meridian when the moon is its furthest north "point M", then arcZM = 2° 7', i.e. the moon is almost at the zenith when it is at its northernmost pass.



Thus at a specific time and date we will have one true position of the moon from the center of the earth (and hence of the universe!) out to the fixed stars. From this one true location of the moon at a given time and from the mean movement tables for the moon which Ptolemy has developed (by methods similar to those by which we developed the tables of the sun's mean movement), we can calculate the moon's true position at any point in time, compare it to the apparent position, and calculate the parallax. So let us presume that we have at hand a table of the moon's mean movement, and a table of the parallaxes it displays at given times.

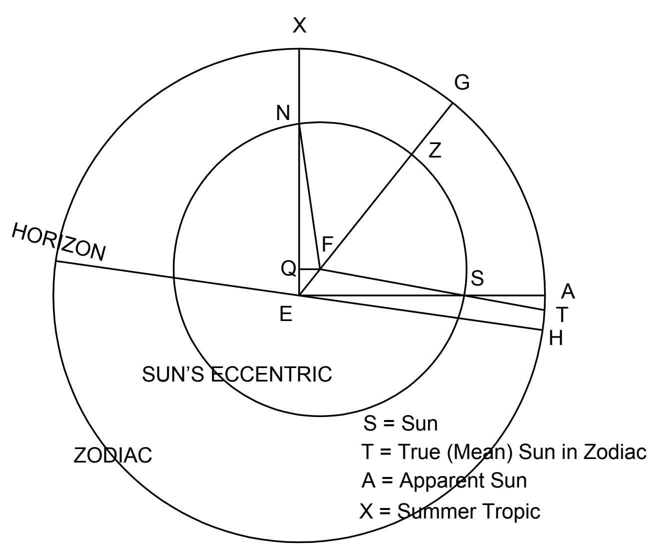
With these preliminaries in place, we can now describe how Ptolemy verified the precession of the equinoxes.

The fixed star which Ptolemy used as a reference is REGULUS, a bright star in Leo which sat pretty much right on the ecliptic. Ptolemy wants to find the angular distance between that fixed star and the summer tropic (also on the ecliptic, of course), and compare that value to one which Hipparchus determined by similar methods about 265 years before him, to see whether it has changed.

As noted above, the technique for this will be to observe the angular distance between the sun and moon on some particular day right around sunset, and then to observe the angular distance between Regulus and the moon about 30 minutes later.

On a particular day, namely "in the year 2 of Antonine, Egyptianwise Pharmouthi 9, as the sun was about to set in Alexandria," Ptolemy observed the moon and sun appearing to be about $92 + 1/8\,°$ apart from one another. Knowing the date and the time, Ptolemy could refer to his tables of the sun's regular movement, and, adding in all the movement around the sun's own eccentric circle (about its own center) from the last time it was known very accurately to be at an equinox or else at a tropic, he could say precisely how far the sun really was from the summer tropic—again, in degrees around its own circular orbit, not in apparent movement around us. He determined that its true position was $93 + 1/20\,°$ west of the summer tropic (or about $3 + 1/20\,°$ into the fishes or pisces).

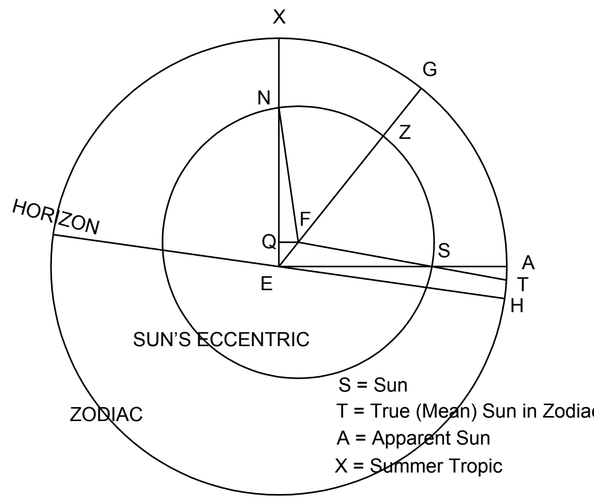
But that does not mean we can simply measure that many degrees west of where the sun was *appearing* and get the location of the summer tropic as it would be appearing to us in our sky (if it would only appear, say, by glowing green out there!). We have to remember that the sun does not appear to us where it "really" is according to its movement around its own center. Let E be earth, H our horizon (or Ptolemy's), A the position in our sky where the sun appears, T the "true" location of the sun as projected onto the celestial sphere from F, the center of the sun's eccentric orbit, and X the summer tropic (a projection of N, the northernmost point in the sun's eccentric orbit, from E, earth). By the date and time and our already established solar theory, we know the angle NFS, i.e. how far the sun is from its northernmost point around F, the center of its own orbit. But from Day 19, we recall that



$\angle FEN = 24°\ 30'$

and EF : FN = 1 : 24

Now there is only one triangle which can have those specifications, and we can determine all its sides and angles trigonometrically, using our table of chords and arcs. In outline, here is how to do that:



• Drop FQ at right angles to EN.
• Since we know ∠EQF is right, and we know ∠ FEQ (i.e. 24° 30'), hence we know all the angles of △FQE.
• Describing a circle on EF as diameter, we know from our table of chords and arcs what EQ and FQ will be in 120th parts of EF. So we know all the sides of △FQE.
• Since we now know the ratio EF : FQ, and we already knew the ratio EF : FN, we also now know the ratio QF : FN, and by the Pythagorean Theorem we also know the ratio of NF to either QN or QF. Hence, by drawing a circle on FN as diameter, we know all the angles at the center of that circle subtending the sides of the triangle as chords, and we can easily determine the angles of △QNF. So that triangle is fully known.
• We know the ratios NF : EF, and QE : EF, and QN : NF. Hence we know the ratio NE : EF. And so the triangle ENF is fully known.
• Hence we know ∠EFN.
• But we know ∠NFS (as we said above).
• So we know ∠EFS (the remainder of 360°).

Knowing the date and time, we also know what the solar anomaly would be at this point. Accounting for the anomalistic difference, we can calculate the value of ∠XES (or ∠XEA), which turns out to be 93°.

So while the sun's true angular distance from the summer tropic at the time of Ptolemy's sunset observation (i.e. ∠NFS) is 93 + 1/20°, the sun's observed or apparent angular distance from the summer tropic at that time (i.e. ∠XES) is just 93°.

So now we know that the visible sun was, as it were, acting as a marker for the apparent location of the summer tropic, i.e., the summer tropic was 93° east of where the sun was appearing.

But the *moon* was appearing 92 + 1/8° eastward of the sun at that particular time. Hence it was appearing

92 + 1/8° = [30° - 3° fishes] + [30° aries] + [30° taurus] + 5 1/8 ° twins

so the apparent moon was 5 + 1/8 ° into the twins at sunset.

Ptolemy notes that this is *close* to 5 + 1/6 ° into the twins which is where his lunar theory says the moon should have been appearing. We'll go with that number, then.

But now we let a half an hour go by, allowing the sun to set and the star Regulus to appear. By the tables of the moon's regular movement, we know that in 30 minutes of time it must have moved eastward along the ecliptic about ¼ °. But that is in true movement, and from its true position 30 minutes ago, not from its apparent position 30 minutes ago. By the tables of the moon's parallaxes, we know that the moon at the time it was observed around sunset on the given date was displaying a 1/12 ° parallax. Adjusting for that, its true position was not 5 + 1/6° into the twins,

but 5 + 1/6 – 1/12° into the twins,

i.e. 5 + 1/12° into the twins.

But knowing its true position at that time, we can now add the regular movement it underwent, namely about ¼ ° in 30 minutes of time after sunset, and so at that point the moon was

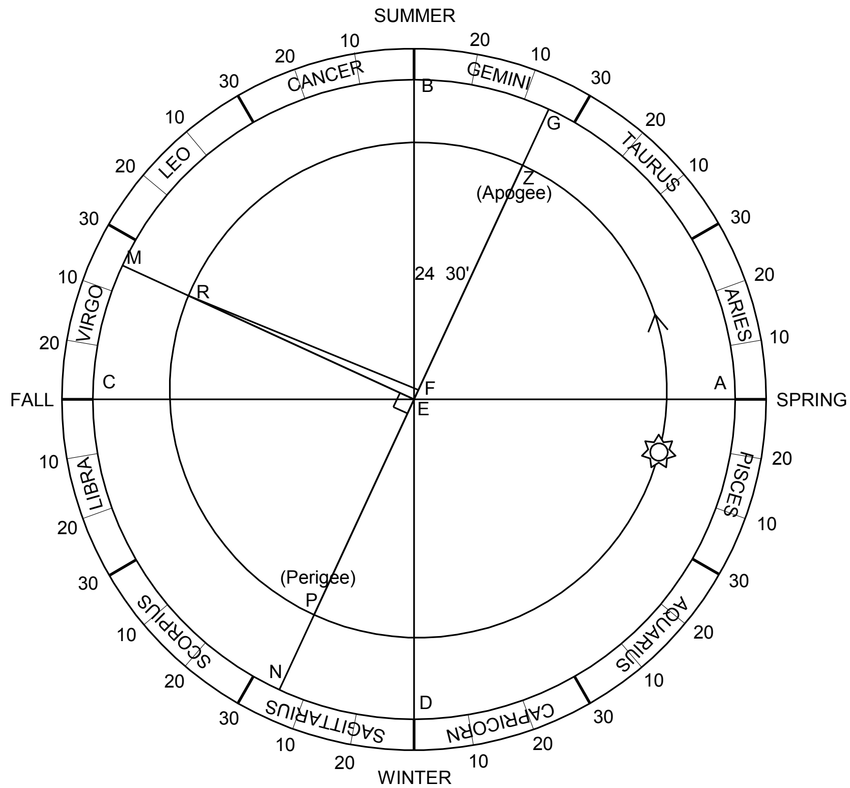
5 + 1/12 + ¼ ° into the twins

i.e. 5 + 1/3 ° into the twins.

Now at that same time when the moon was appearing 5 + 1/3° into the twins, the star Regulus appeared 57 + 1/6° eastward of the apparent moon. Hence Regulus was appearing 57 + 1/6° eastward of 5 + 1/3° into the twins, i.e. 57 + 1/6 + 5 + 1/3° east of the beginning of the twins

i.e. 62 ½ ° east of the beginning of the twins,

i.e. 32 ½ ° east of the beginning of cancer, the summer tropic for Ptolemy.



So in the time of Ptolemy, Regulus was 32 ½ ° east of the summer tropic.

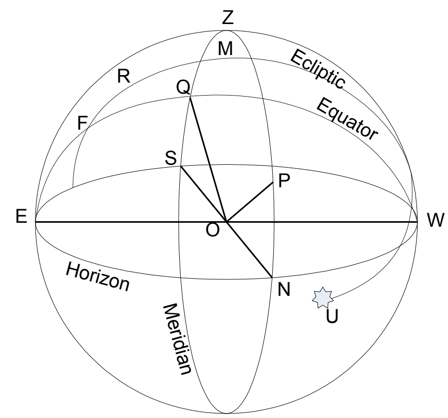
But Ptolemy has the records and calculations of Hipparchus, his predecessor who made similar observations 265 years earlier, according to whom Regulus was in his time 29 + ½ + 1/3 ° east of the summer tropic.

Therefore, in 265 years, the zodiac moved 2 2/3 ° east, relative to the 4 points on the ecliptic.

Therefore the equinoxes precess about 1° every 100 years.

The modern figure is 50.26’’ a year = 1° 23’ 46’’ every 100 years, so the period is about 25,800 years.

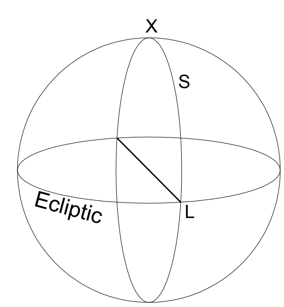
The adjacent diagram illustrates the situation in Alexandria, year 2 Antonine, 9th day of month Pharmouthi, 5:30 pm, when Ptolemy made his observations by which to determine the distance between Regulus and the summer tropic.



O = Observer
P = Polaris
Z = Zenith
M = Moon, on the intersection of the ecliptic and the meridian, culminating in Twins (5 1/6 ° into Twins, 92 1/8 ° east of sun)
U = Set sun, 30 minutes after sunset, about 3 1/20 ° into Fishes
R = Regulus in Leo, 57 1/6 ° east of moon, and on the ecliptic
F = Fall equinox

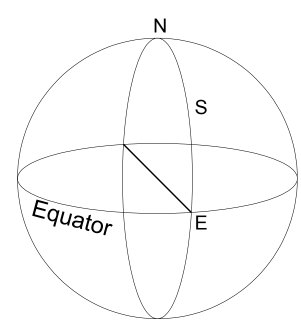
PRECESSION IS ABOUT THE POLES OF THE ECLIPTIC.

Now let’s develop a clearer image of this motion.



Looking at the three adjacent diagrams,
Let X = a pole of the ecliptic
and N = the north celestial pole
and P = Polaris

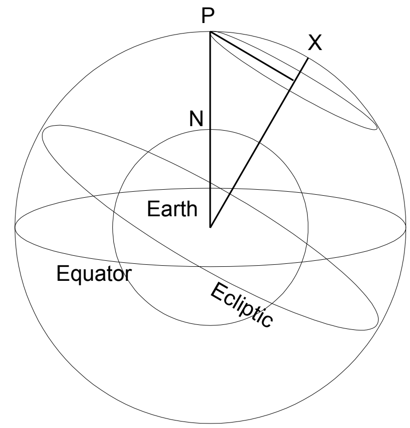
For any fixed star S, the great arc XSL (down to the ecliptic) has the same value all the time, 24 hours a day, 10 centuries a millennium (strictly speaking even this is not true, since the stars do drift around; but they do so too slowly and imperceptibly for us to care about for these purposes; ten or twenty thousand years from now the “Big Dipper” will not look much like a dipper, but that is impossible to verify with naked eye observations over a few years or even over a hundred years).



But the great arc NSE (down to the equator) changes a bit over great lengths of time.

The ecliptic is therefore a more permanent and principal line on the sphere of fixed stars than the equator is!

Therefore the *earth's* north pole is more abiding than the *celestial* north pole! (Shouldn't that give us a clue? Doesn't that hint that it is not the heavens that are moving, but the earth, and that is why the place called North Pole on earth always stays the same, but the spot in the sky called North Pole does not?)



And since the great arc XSL always stays the same, that implies that the precessional movement of the equinoxes is about the poles of the ecliptic.

WHERE ARE THE SOLSTICES & EQUINOXES TODAY, compared to where they were for Ptolemy?

In Book 7 Chapter 2, Ptolemy says he observed Regulus at 32 ½ ° from the Summer Tropic, 2 ½ ° into the Lion. Thus the summer tropic was 2 ½ ° of Lion, plus 30° of Crab *west* of the star Regulus, and therefore was at 0° Crab. If Ptolemy was about 50 years old at the time of this observation, it was about 150 A.D.

In the same chapter, Ptolemy says Hipparchus observed the summer tropic 265 years earlier [about 115 B.C.] only 29.83° west of Regulus, and since Regulus is 2 ½° into the Lion, he saw the summer tropic at (29.83° – 2 ½ °) = 27.33° westward into the Crab, i.e. it was at (30° – 27.33°) = 2.67° Crab.

A current value for this precession is 50.26'' a year, or .013961111° a year, with a period of about 25,800 years.

Now take the current year, and subtract 150 from it, and you have about how many years elapsed since Ptolemy's observation. I will use 2012, the year in which I am writing this (you can figure it out yourself for the current year, of course.) Then 2012(.013961111°) = the number of degrees of precession since Ptolemy up to 2012 = 28.09°. Now the spring equinox was at the beginning of the fishes (or Pisces) for Ptolemy. It has moved almost 30° eastward since his time, which means it is almost in the water-bearer, i.e. at the beginning of Aquarius. So a little more westward precession, and we will have the "dawning of the age of Aquarius," when the spring equinox passes out of Pisces and into Aquarius. Whoopee!

WHO CARES, ANYWAY? Astrology aside, what is interesting about the precession of the equinoxes?

(a) Historically, with Hipparchus and Ptolemy, people for the first time realized that the movement of the sphere of fixed stars is not perfectly uniform. Aristotle thought it was. So did Plato. Aristotle in some way thought that the motion of that sphere was time itself, being the fastest and most uniform motion. The sphere was the "primum mobile," moved by God. But now there is an irregularity in it, or at least a composition of motions in it. This new fact had implications for Aristotle's *Metaphysics*, which culminates in certain arguments about the gods on the supposition that the sphere of fixed stars has a single, perfectly uniform and eternal motion. Later medieval thinkers posited further spheres outside the sphere of fixed stars, supposing that the outermost of these had a single and perfectly uniform motion, and that it was the true "primum mobile."

(b) Astronomically, this new fact suggests we might have something wrong. It is difficult to believe that the whole universe is moving around us, but it is far more difficult to believe that it has a complex motion, rather than a perfectly simple and uniform one. Stranger still, not only does it keep changing the fixed points about which it rotates daily (i.e. the celestial poles), but it takes great pains to ensure that whatever these points are at the moment, they hover directly over the same two spots on earth—our north and south poles. Why would the heavens go to all that trouble?

(c) Physically it poses an interesting problem. Once we get to Copernicus, we see that the precession must be explained as a kind of wobble in Earth's axis. But why does that happen? Newton was the first to explain it—it is a phenomenon of differential gravitation, due to the Earth's not-perfectly-spherical shape (it is an "oblate spheroid") and due to the Sun's (and Moon's) consequently uneven attractions of the Earth.

PTOLEMY

DAY 21

INTRODUCTION TO THE THEORY OF THE PLANETS

We have seen some of what Ptolemy had to say about the fixed stars. He had more to say than what we discussed—for example, he catalogued the constellations of the northern and southern hemispheres.

But he had far more to say about the wandering stars, the planets. That is partly because they are closer and brighter and easier to observe, but more because they have special movements among the stars. They do not keep a fixed position among the "fixed" stars, but wander through them over time. Like the sun and the moon, they appear to creep eastward through the stars, all of them keeping within 8° or so on either side of the ecliptic. Unlike the sun and moon, however, the planets have other anomalies in their apparent movements. They appear to stop sometimes—this is called a "**station**"—and then when they start moving again, they can appear to move backwards, that is, they appear to creep *westward* rather than *eastward.* That is called "**retrogradation**" or "retrograde motion" or "regression." Then they exhibit another station (a westward station), and then begin "**progression**" eastward once again! So the planets make zig-zags in the sky. Mars, for instance, will go through this pattern:

- moving eastward through the stars (progression)
- stop for a while (eastern station)
- moving westward through the stars (regression)
- stop for a while (western station)
- moving eastward through the stars ...

Moreover, exactly how often they do this, and for how long (i.e. how great a regression), seems to be tied to the sun. Mars, Jupiter and Saturn, for example, seem to go through their stations and retrograde motions when they are near "**solar opposition**", that is, which is one a planet and the sun are on exact opposite sides of the earth, 180° apart in the zodiac. Also, the size of their retrograde motion seems to depend upon where it takes place in the zodiac. But more on these details later.

We are skipping ahead to Book 9 of the *Almagest*, now, where Ptolemy begins his planetary theory in earnest.

BOOK 9 CHAPTER 1

THE PTOLEMAIC VIEW OF THE WHOLE.

Before getting into the details of planetary theory, Ptolemy wants to give us a sense of the whole.

He is unaware of Uranus, Neptune, and Pluto (now demoted from planet-hood), since these cannot be seen with the naked eye. So he has 5 planets to explain and order, besides the sun and moon: Venus, Mercury, Mars, Jupiter, and Saturn.

But he is not 100% sure about the order of their spheres around us. (Just as the fixed stars are on a "sphere" whose spinning causes their circular motions, so too he conceived of each planet, and the sun and the moon, as embedded in or on a perfectly transparent sphere—or even embedded in a sphere on a sphere, so as to produce the epicyclic motions.) He knows that the moon is the lowest of all, closest to us. One reason for this is that he is able to get parallax on the moon, but not on any of the other heavenly bodies. Another reason is that the moon sometimes comes between us and the sun (during a solar eclipse), but the sun never comes between us and the moon. Similarly, the moon comes between us and the fixed stars and planets, blocking our view of them, but they never come between us and the moon. But after that it is not so easy to say! Which is closer to us, Mars or Venus? Mercury or the Sun? Jupiter or Saturn?

He does not make the order among them a priority in his work. If he had, he might have started to move more toward the Copernican view, the heliocentric view. More on that when we come to Copernicus. Still, he takes a stab at the order of the planets, going out from us, the center of the universe.

THE PTOLEMAIC ORDER OF THE SPHERES.

The order of celestial bodies (going outward from Earth) according to Ptolemy is this:

Earth
Moon
"Inner Planets"
Sun
"Outer Planets"
Sphere of Fixed Stars

The "inner planets" are Venus and Mercury. He is not entirely sure about the order of their spheres, but he seems to think the order is Venus first (right after the moon), then Mercury is higher up and further out.

The "outer planets" are Mars, Jupiter, and Saturn, in that order. All significant astronomers of Ptolemy's time agreed that this is the order of the outer planets, that they are further from us than Venus and Mercury, and that Venus and Mercury are further than the moon.

WHERE IS THE SUN?

But there was disagreement as to where the Sun falls.

Among Ptolemy's predecessors, EARLIER mathematicians said it falls between the inner and outer planets, while LATER mathematicians said it falls between the moon and all the other planets.

In favor of the LATER ones, we never see the sun eclipsed by the inner planets (but the fact is they do pass in front of it; those ancient astronomers just could not observe their solar transits without a telescope and some special filters).

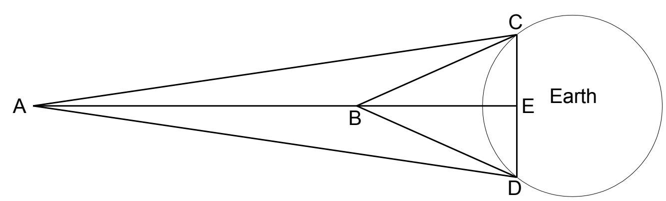
In favor of the EARLIER ones (whose opinion Ptolemy prefers), there is a natural division between the 2 planets that always stay near the sun on the ecliptic, and the 3 that don't. So he wants to put the sun between them.

NO CLEAR IDEA THAT THE PLANETS CIRCLE THE SUN.

Ptolemy seems to allow for the possibility that the inner planets make epicycles which do not encompass the sun, but are just closer to us than the sun.

This is odd, since they go back and forth on either side of the sun, strongly suggesting they encompass the sun!

THE PTOLEMAIC UNCERTAINTY.



He can't be sure about his order of distances because no "parallax" is detectible to the naked eye.

"Parallax", as we noted back in Day 20, is the difference in where a thing appears against some background from two different locations. Parallax is useful for determining relative distances (e.g. on a background of fixed stars, the planet with the greater parallax is closer). (Try opening and closing alternate eyes while looking at two of your fingers held at different distances from your eyes, and note how much each one seems to "jump" with reference to the wall beyond it.)

If 2 bodies, A and B, are sufficiently close to earth, and locations C and D are sufficiently far apart on the earth, then (as long as observers at C and D can make simultaneous observations of A and B) we will be able to get parallax on both bodies, and by seeing which is greater we will know which is closer. Comparing notes, observers C and D might determine that their simultaneous observations of A and B find them appearing at different places among the fixed stars. Now the line from D to B is really from D to a fixed star against which B appears, and the line from C to that same fixed star will be parallel to DB (since the earth is as a point to the heavens—see Day 2). So the angle between CB and the line from C parallel to DB is known. But that is equal to the angle CBD, the angle of parallax. If this is done for two different bodies, A and B (and they do not have to be observed at the same time—just the observations of B must be simultaneous, and again those of A must be simultaneous, but there is no need to observe A and B simultaneously), and the angles of parallax are compared, the one with the greater angle of parallax is nearer to the earth.

But if bodies A and B (for example) were too far away, CA would be sensibly parallel to DA, and CB to DB, and we get no triangles to work with, and no angles of parallax to compare.

(Parallax on Mars was first obtained by Cassini in the time of Newton, and that was the first time, historically, that we were able to get absolute distances to the heavenly bodies. But we obtain the relative ones before that, even with Kepler. More on these things later.)

CHAPTER 2

THE GENERAL GOAL OF PLANETARY THEORY.

In Chapter 2 of Book 9 of the *Almagest*, Ptolemy explains the goal, the necessity, the difficulty, and the mode of proceeding of his planetary theory to follow.

(1) The GOAL is to explain the double anomaly of the planets accurately, i.e. to produce "astronomically-correct" (uniform circular motion only, please!) models which will produce exactly what the appearances are.

(2) The NECESSITY of doing so is due to the fact that it is really worthwhile and no one prior to Ptolemy had yet succeeded.

(3) The DIFFICULTY of doing so is due to the fact that there are 2 anomalies going on at once for each planet "so it is very hard to determine what belongs to each", and again because we need a long record of observations, and very accurate ones, but many recorded observations are not very accurate, but are "thrown together carelessly and grossly", and many things are difficult to observe accurately, such as the stations (the exact moment they occur).

(4) The MODE of proceeding he will adopt is accordingly as follows:

(a) He will use Hipparchus's observations, because Hipparchus was a serious astronomer, no slouch. In general, he will use "only those observations which cannot be disputed," e.g. when a planet was right on top of a fixed star, so its locale was absolutely clear.

(b) He will use certain oversimplifications, e.g. assuming things are all in one plane, when there are slight latitudinal differences, so long as these simplifications are not too false to make a big difference.

(c) He will use certain assumptions suggested by trial and error but impossible to prove by some simple observation, i.e. he will allow things whose consonance with appearances will be clear, although how we discovered them is not at all clear.

(d) He will use models with properties as demanded by the appearances, i.e. he will not demand total uniformity of model from planet to planet, e.g. that every epicycle should have the same tilt relative to its deferent.

THE DOUBLE ANOMALY.

Ptolemy does not describe this clearly and succinctly. Ptolemy only says that one anomaly is tied to the sun, the other to the zodiac—hence the names "heliacal anomaly" and "zodiacal anomaly."

But the zodiacal anomaly is an inequality or non-uniformity in the heliacal, so the heliacal must be understood first. And what the heliacal is for inner planets is not the same as for outer ones, exactly! So there are in a way 4 kinds of anomaly, 2 for each kind of planet (inner and outer), and they are analogous but not entirely identical. Let's describe these anomalies now, since the whole point of what is to follow will be to find models which will explain these anomalies in the apparent movements of the planets.

DISTINCTION BETWEEN INNER AND OUTER PLANETS.

The "inner planets," Mercury and Venus, are never found more than a certain angular distance from the sun. They have maximum or "**greatest elongations**" from the sun, either on the western side or on the eastern side of the sun. By contrast, the "outer planets," Mars, Jupiter, Saturn, can be any angular distance from the sun. This is the big distinction between inner and outer planets.

(The cause of this, in truth, is clear. Our own orbit encompasses those of Venus and Mercury, while it is encompassed by those of Mars, Jupiter, and Saturn. This is another sneak peek at Copernicus. For many of us, THEORY is more familiar than APPEARANCES, since we have been told certain things all our lives; so we can often work from the theory back to what the appearances must be, if we don't know or can't remember.)

Mercury's greatest elongations from the sun are smaller than those of Venus, which gets a little further from the sun. Incidentally, this makes Mercury a pain to observe. It is so close to the sun that it is generally visible only right after sunset or just before sunrise (generally when the sun is up it is too bright to see Mercury with the naked eye).

Again, all 5 planets have ROUGHLY the same orbital plane, i.e. they move in longitude pretty much around the ecliptic (or in the zodiac, the fat version of the ecliptic made to contain their orbits).

Obviously, if Venus & Mercury are sometimes tied to the sun, sometimes on one side of it, other times on the other, then they must be seen at times ahead of and at other times behind the sun on the ecliptic—hence they are MORNING STARS and also EVENING STARS:

Eastern Elongation = Evening Star (planet is seen after sunset)

Western Elongation = Morning Star (planet is seen before sunrise)

THE HELIACAL ANOMALY FOR INNER PLANETS, then, is just this: since they get ahead of and behind the Mean Sun (the Mean Sun is never further away from the aApparent than the solar greatest anomalistic difference, which is only 2° 23', whereas Venus can get as far away from the Mean Sun as about 49°), therefore *they speed up and slow down* in their motion around us, which is an anomaly.

It is called "HELIACAL" because it is a motion back and forth on either side of the Mean Sun—that is the uniform motion around us which the irregular inner-planetary motions seem to be tied to. They can speed up and slow down compared to the Mean Sun, but only by so much, and cannot get too far away from it in speed or distance.

In fact, what is the AVERAGE SPEED of Venus going to be around us? That of the Mean Sun, of course!

And what is the AVERAGE SPEED of Mercury going to be around us? That of the Mean Sun, of course!

But each gets ahead of and behind it in a periodic way, hence speeding up and slowing down, and reaching a GREATEST ELONGATION from the Sun, then disappearing into it (in front of or behind it, in truth!), and appearing again on the other side and making an opposite greatest elongation.

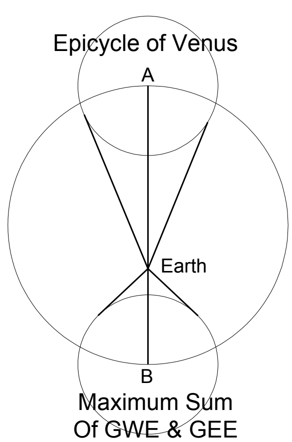
So it goes from GWE (Greatest Western Elongation) to GEE (Greatest Eastern Elongation) and back again.

And ONE CYCLE of the heliacal anomaly is just that: the movement (or time) from one GWE back to the next GWE, or else from one GEE back to the next GEE.

Note that this cyclic motion right away suggests that Venus is on an EPICYCLE. (Or, more correctly, it suggests that Venus is orbiting the Sun!)

And this epicyclic motion happens to be of the right speed and direction that Venus also appears to make stations, and to move backwards sometimes in the zodiac. But since it has greatest elongations, these are more telling and striking, and we work out our Venus theory mostly from those.

ZODIACAL ANOMALY FOR INNER PLANETS.



So the time from one GEE to the next GEE is one cycle of the heliacal anomaly, which takes more than 1 year for Venus, namely about 19 months.

ZODIACAL ANOMALY: But it turns out there is not just one value for a GEE (or GWE) for Venus. This value is bigger or smaller depending on *where the sun is on the ecliptic when Venus reaches its maximum elongation*. The reason for this (we hypothesize) is that Venus's epicycle (or, cheating, its orbit around the sun) itself moves on a circle (deferent) around us but we are not at the center of that circle, and so Venus's epicycle *appears larger or smaller* from where we are, depending on where in the ecliptic it is appearing. So the SUM of the GEE and GWE will be the same, for Venus, when the mean sun is at a given point in the ecliptic, but that sum will vary for different spots in the zodiac. And the maximum sum is directly opposite where the minimum sum occurs, in the ecliptic.

This variation of the apparent heliacal anomaly is called the "**zodiacal anomaly**."

And ONE CYCLE of the zodiacal anomaly WOULD be the time from a greatest elongation in some part of the zodiac back to another greatest elongation (of the same kind) at that same place in the zodiac, except that the next time it is in that part of the zodiac the planet might not happen to be at its greatest elongation from the sun. In fact, if the speeds on the epicycle and on the deferent are incommensurable, you will never exactly get another greatest elongation (of the same kind) in that part of the zodiac! So "one cycle of zodiacal anomaly" means the same thing as "one time of the epicycle around the eccentric deferent".

HELIACAL ANOMALY FOR OUTER PLANETS.

For the outer planets (which are tied to the movement of the sun in a different way, although they can get any angular distance away from the sun):

HELIACAL ANOMALY = when almost directly opposite the sun, the planet stops its eastward motion in the sky, and we get a station. When the sun becomes even more directly opposite, the planet appears to move westward in the sky, a retrogradation. As the sun moves on, the planet again appears to stop, and we have another station, and finally it moves eastward once more. So it makes a ZIGZAG in the sky, whenever it goes through solar opposition. This also strongly suggests an EPICYCLE. The planet appears to zig-zag, and also to stop at times, because the movement of the planet on its epicycle sometimes cooperates with, sometimes fights against, its general motion around the deferent—and sometimes the two motions exactly cancel each other out in the appearances to us, and hence we get a station.

The time from a first station (prior to retrogradation) back to the next corresponding station (prior to next retrogradation) is ONE CYCLE of heliacal anomaly; for example, the time from an eastern station to the next eastern station.

ZODIACAL ANOMALY FOR OUTER PLANETS.

The ZODIACAL ANOMALY for outer planets refers to the fact that the apparent speed of retrogradation, and how long it takes, and how great a length in the sky for which it retrogrades etc., varies for different positions of the mean sun on the ecliptic. So again, this suggests an eccentric deferent.

CHAPTER 3

Recall that the way we gained insight into the Sun's eccentric path, and the way we determined where it "really" is on that path around its own center, was first of all by determining the period of the sun's eastward motion. We need to do something similar with the planets. We need to have some notion about what regular motions we think they are accomplishing on certain circles, and then a way of determining the periods of those motions. To that end, we will distinguish "mean movement in longitude" from "mean movement in anomaly", and we will determine the "least periodic joint returns", for each planet.

WHAT ARE MEAN MOVEMENTS IN LONGITUDE & MOVEMENT IN ANOMALY?

Mean Movement in Longitude = uniform motion of epicycle's center on deferent
Mean Movement in Anomaly = uniform motion of star on epicycle

Note: Since the deferent is going to be eccentric for each planet, the mean motion of the epicycle in longitude is measured around the center of the deferent, not around us. (Actually, we will soon introduce still another point around which equal angles are swept out in equal times, the equant.)

WHAT IS A "LEAST PERIODIC JOINT RETURN"?

This means the shortest time required for some whole number of cycles of anomaly AND some whole number of cycles of longitude to be completed together.

Let's take an imaginary example. Suppose planet P is moving on an epicycle, and the period of its own motion around the epicycle is 3 years. Every 3 years it goes around its epicycle exactly once. But the period of its epicycle around its deferent is 5 years. Every 5 years its epicycle goes around the deferent exactly once. Well, then, starting at any precise moment, when the star is someplace on its epicycle and the epicycle is someplace on the deferent, how long will it be before the star and epicycle are both back in those same places again? 15 years. In that time, the star will have gone around exactly 5 times (since its period is 3 years), and so it will be back at the same place on the epicycle, and again the epicycle will have gone around exactly 3 times (since its period is 5 years), and so it will be back at that same place on the deferent. But in any shorter amount of time, at least one of the two motions will not be back at the same spot. We are looking for a least common period, as it were, of the two motions.

Since 15 is the least common multiple of 3 and 5, this technique will work so long as there is a common multiple of the two periods. But if the two periods have no common multiple, e.g. $\sqrt{3}$ and 5 do not, then there will be no exact least periodic joint return, although if you wait long enough you can get as near as you please to a joint return. So if the 2 periods are *INCOMMENSURABLE*, and the two motions begin at some place together, they will never again be at those same two points at the same time, lest the speeds become commensurable. And where is it written that the period in longitude and in anomaly must be commensurable? Well, that is why Ptolemy says "that is, VERY NEARLY JOINT." We get some decent whole-number approximation.

If in the same time the two movements return back to their same starting points, then each accomplishes a whole number of cycles in the same time, and hence their speeds would have to be commensurable. So if incommensurable, they never line up again in the position they were in when they both began. They never have the same relative position twice.

WHY DO WE CARE ABOUT THESE?

Recall that the motion of the planets appears irregular, but we believe it is truly regular and circular, or composed of such motions. How do we get a hold of their regular motions, if they are appearing to move irregularly? By getting their PERIOD on this or that circle (which circle is one that we hypothesize based on the suggestiveness of the appearances, e.g. an epicycle).

That is, we cannot observe the uniformity at all times, but if we suppose there really is uniformity of motion, then we know that the time from HERE back to the SAME PLACE on its circle must always be the same, and that will give us the UNIFORM SPEED (in angular velocity).

So we just need to know when a planet is "back to the same place," e.g., on its epicycle, or when the epicycle is back to the same place in the zodiac.

But here's the problem: We cannot see the center of the epicycle, or anything at all except the star itself, which appears to move irregularly. So how do we know, for instance, that "the star is back where it was on the epicycle"?

We can (for Venus) measure the time from GEE to the next GEE, and this might tempt us to say "there, the star moved 360° in that time, so we know the period in anomaly." But did it really go 360°? Not quite, since at the next GEE it was closer to us, or further, which changes where the POINT OF TANGENCY hits the epicycle.

(Similarly for stations for the outer planets.)

So what we really need is to find when the star is not only at a GEE, but when it is at a GEE *back in the same part of the zodiac*, which means not only is the star back at the same place on the epicycle, but also the epicycle is back at the same place on the deferent; i.e., we need a JOINT RETURN.

PTOLEMY

DAY 22

PERIODIC JOINT RETURNS; THE EQUANT

We want to get the least periodic joint returns for the 5 planets—that is, we want to be able to say, for a given planet, the number of times the epicycle goes around the deferent in the same time that the star goes around the epicycle some whole number of times (and we want to express this in least numbers). We will put up with some slight inequality, though, since the two speeds (of star and of epicycle) might be such that they never complete whole numbers of cycles in exactly the same time (as we noted in Day 20), or else they might do so, but the smallest number of times the epicycle goes round in the same time that the star completes some whole number of cycles on the epicycle is 6 trillion. We don't have time to wait around for that! So Ptolemy instead gives us "very nearly joint returns" for the 5 planets, as follows:

	SUN # times mean sun orbits earth	LONGITUDE # times epicycle's center orbits deferent	ANOMALY # times star orbits epicycle
Saturn	59 (+ 1 45/60 days)	2 (+ 1 43/60 °)	57
Jupiter	71 (– 4 54/60 days)	6 (+ 4 50/60 °)	65
Mars	79 (+ 3 13/60 days)	42 (+ 3 10/60 °)	37
Venus	8 (– 2 18/60 days)	8 (– 2 15/60°)	5
Mercury	46 (+ 1 2/60 days)	46 (+ 1°)	145

In other words, during the time it takes the mean sun to go around us 59 times (plus a little bit), Saturn's epicycle goes around its deferent 2 times (plus a tiny bit) and Saturn itself goes around its epicycle 57 times (precisely). And so on with the other entries.

What kinds of observations would Ptolemy have had to make in order to derive these numbers? In the case of an inner planet, like Venus, he could look at tables of data (of his own, or from Hipparchus and others) correlating where Venus was appearing in the zodiac and the date and what its elongation from the mean sun was at the time. If we look at such tables, we will notice that the value of Venus's greatest elongation from the sun varies depending upon where it is in the zodiac. And it is not the case that if it is at a greatest western elongation from the sun in place X in the zodiac, then a year later, when it is at that place in the zodiac, it will again be at a greatest elongation. Rather, we have to wait 8 years. Then it will be at a greatest western elongation again, and in pretty much the same part of the zodiac, and the size of that elongation will be what it had been 8 years before. But during that time, we can see that the mean sun went around us 8 times (of course), and so Venus's epicycle, which ties Venus to the mean sun and hence itself has the same speed as the mean sun also went around us 8 times, and we can count the number of times that Venus went from greatest western elongation back to greatest western elongation in those 8 years, i.e. its cycles of anomaly, namely 5 times. Let that suffice for an understanding (somewhat oversimplified) of how to derive the table of periodic returns above from the raw observations.

NOTE THE OTHER PATTERNS: S = L, and S = L + A.

If we get rid of the "plus a little" or "minus a little" in the table, and present a simplified version of it, we have:

	<u>SUN</u> # times mean sun orbits earth	<u>LONGITUDE</u> # times epicycle's center orbits deferent	<u>ANOMALY</u> # times star orbits epicycle
Saturn	59	2	57
Jupiter	71	6	65
Mars	79	42	37
Venus	8	8	5
Mercury	46	46	145

We have already seen that the number of solar cycles, S, equals the number of longitudinal cycles (times the epicycle goes round), L, for inner planets. But what about the outer planets? Do you notice something about their numbers?

59	=	2	+	57
71	=	6	+	65
79	=	42	+	37

For the outer planets, for some reason, the number of solar cycles is equal to the sum of the number of longitudinal cycles plus the number of cycles of anomaly—in other words, $S = L + A$. A remarkable coincidence!

For the inner planets, too, it is a coincidence that $S = L$. There is no reason why these things should be so in Ptolemy—they just are. They are cosmic coincidences which need not be. But Ptolemy is aware of them. He is so aware that $S = L$, for instance, that the entry on the table for Venus's mean movement in longitude (i.e. the motion of the center of its epicycle) in 1 day is exactly the same value as that for the mean sun. In fact, that whole table is just the table of the mean sun's movement reproduced.

$S = L$ follows for Ptolemy only because the inner planets are each tied to the sun, and hence their epicycles must have the same average speed as the sun, i.e. the speed of the mean sun. But why should that be, if they do not orbit the sun, but make epicycles in front of it, closer to us?

$S = L + A$ is even more arbitrary-sounding right now. But there it is!

GENERATING THE TABLES OF PLANETARY MEAN MOVEMENTS IN LONGITUDE AND ANOMALY.

As we did with the sun, so we do now with the planets. If you tell me how far the epicycle moves in longitude in any amount of time, then, since I know it is uniform, I can tell you how far it moves in any other time, or, conversely, given how far it has moved, I can tell you how long it took.

So too with the mean motion in anomaly, i.e. the uniform motion of the star on the epicycle.

CHAPTER 5

DETERMINATION OF THE GENERAL PLANETARY HYPOTHESIS.

We have already seen that the planetary phenomena are highly suggestive of epicycles on eccentric deferents. And we have also seen how to determine the relative speeds of the star on the epicycle and of the epicycle on the deferent by means of the least periodic joint returns. But are we talking about same-direction epicycles, or opposite-direction epicycles? Do the star and epicycle rotate in the same directions about the centers of the epicycle and deferent respectively? Or do they go in opposite directions?

If we are to explain the planetary phenomena by epicycles on eccentric deferents, we must do so by means of SAME-DIRECTION EPICYCLES. The following explanation will bring out the reason for this.

First, recall that the "greatest passage" of a star means its fastest apparent speed, or the moment dividing the time in which it was speeding up from the time in which it will be slowing down, while the "least passage" of it means its slowest apparent speed, or the moment dividing the time in which it was slowing down from the time in which it will be speeding up. The "mean passage" of a star means when it appears to be moving with its mean speed (e.g. with the uniform speed which its epicycle actually has around the center of the deferent), or, alternatively, the moment dividing the time in which it had a speed less than the mean speed from the time in which it will have a speed greater than the mean speed (or else the moment dividing the time in which it had a speed greater than the mean speed from the time in which it will have a speed less than the mean speed).

Let G = Greatest passage
M = Mean passage
L = Least passage

In the case of all 5 planets, for the heliacal anomaly,

Time [G to M] > Time [M to L]

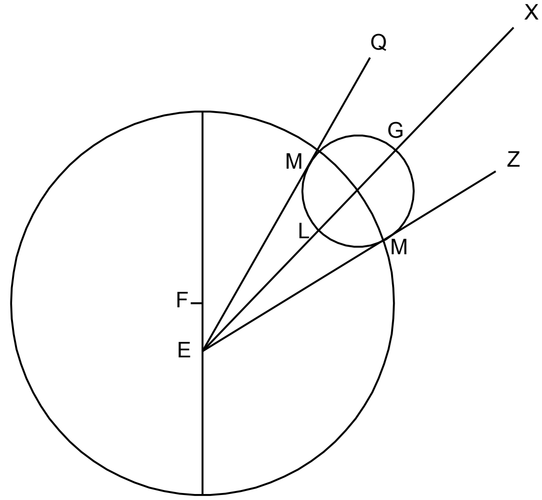
and therefore we are dealing with a SAME-DIRECTION EPICYCLE, where greatest passage is at apogee.

In the case of all 5 planets, for the zodiacal anomaly,

Time [L to M] > Time [M to G]

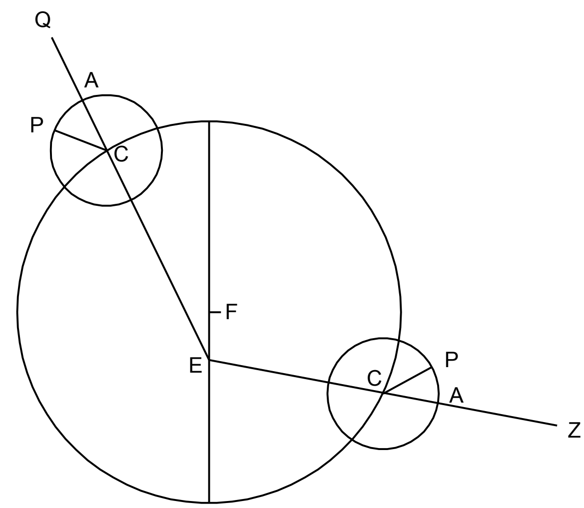
and therefore the ECCENTRIC (deferent) will do to explain that anomaly.

Ptolemy is very terse about the observations which justify these claims. The strange thing is that he is separating the appearances for the two anomalies of a single planet, whereas a planet has only one set of appearances, not two. It seems he is assuming that the planet is moving on an epicycle, and then isolating the appearances due to the planet's motion on the epicycle, and again isolating the appearances due to the epicycle's motion on the deferent. Let's see briefly how he does this for the heliacal and zodiacal anomalies.



ISOLATING THE HELIACAL ANOMALY. At some time the planet is seen making a greatest passage when it is against some spot, X, in the zodiac. Some whole number of cycles of longitude later, the center of the epicycle, C, is back at the same spot—and we can know when that is the case, since we know the period of C around the center of the deferent, F, thanks to our periodic joint returns. Very well, the next time we know C is back at the same spot it was in when we observed the planet in greatest passage against X in the zodiac, we observe the planet again, and this time we see it moving with its mean speed (which is the same as the speed of C around F, which we know by the periodic joint returns). When it is at mean passage, it is viewed against Q in the zodiac. We do this again still later, and we observe the planet moving with its least speed, and it is once again viewed against X in the zodiac. Since we are viewing the apparent speed of the planet (over a couple of nights, so that the epicycle does not move much) for one location of the epicycle, the effect of the zodiacal anomaly is removed, and all the differences in apparent speed are due to the direction in which the planet is moving on the epicycle. And although the planet does not go from its greatest passage to its mean passage without the epicycle moving meanwhile, we know the rate at which the star really moves on the epicycle (thanks to the periodic joint returns again). Hence after observing the planet at greatest passage on one date, and then at mean passage at another, and then at least passage at still another (at which dates C is back at the same spot on the deferent), it is just a matter of number-crunching to see that the time from greatest to mean is greater than the time from mean to least. And that is a property of the SAME-DIRECTION EPICYCLE. Hence we must employ that simple hypothesis in order to explain the heliacal anomaly.

ISOLATING THE ZODIACAL ANOMALY. When the center of the epicycle, C, is "appearing" at some place in the zodiac, Z (although we can't really see it), let the planet be at P on the epicycle. Observe its speed (over a few nights chart its longitudinal progress or regress). When will the planet next be at that same spot on the epicycle, P? We know the answer from our tables of the planet's regular movement in anomaly (i.e. on its epicycle), thanks to our periodic joint returns. When that time has elapsed, we observe the planet's speed again. We keep doing this, and soon (well, actually after a long time) we have a table of the planet's apparent speeds throughout the zodiac when it is on a certain point P on the epicycle. That means all the differences in its apparent speeds will be due not to its motion on the epicycle (which has been removed from these considerations) but to some other irregularity in its motion around us. Doing as we did with the heliacal anomaly a moment ago, we find that the time from least passage to mean passage is always greater than the time from mean to greatest. This means we can account for this apparent irregularity of speed by an eccentric circle. So we place our same-direction epicycle upon an eccentric deferent.



NOTE: Any slight inaccuracy in our table of mean motions is too slight to be relevant to the crude inequality Ptolemy needs, i.e. concerning the time [L to M] and time [M to G].

DIFFICULTIES. THE EQUANT.

But things won't be so simple as an eccentric deferent! Ptolemy notes this in Book 9 Chapter 5. There are two principal complications which will force us to make our basic model for planetary motion a bit more sophisticated:

[1] The lines of apsides for the planets precess eastward with the speed of the precession of the equinoxes (here's a bit of a spoiler: this phenomenon which Ptolemy notes is just caused by the rotation of earth's axis, too. Real precession of perihelion for a planet completes a back-and-forth cycle once every 100,000 years or so, and is a small oscillation. The planetary apehelia are basically at rest; so much so that Newton refers to "the resting of the aphelia" as an astronomical phenomenon).

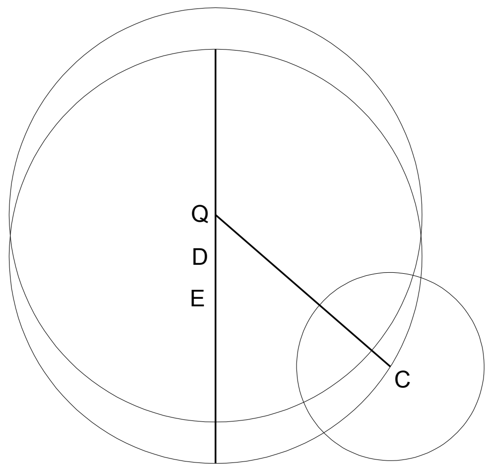
[2] The epicycle for each planet is carried around a circle with one center, but the center of uniform motion is another point (which I will call the "equant" or equalizing point).

As Ptolemy puts it: The epicycles' centers are borne on circles equal to "the eccentrics effecting the anomaly" [i.e. the equant circles] but described about "other centers" [i.e. the centers of the eccentric deferents], and these "other centers" [i.e. the centers of the eccentric deferents] in the case of all except Mercury [which is more complex] bisect the straight lines between the "centers of the eccentrics effecting the anomaly" [i.e. the centers of the equant circles] and the center of the ecliptic [us].

That is characteristically obscure of Ptolemy. Let's see if we can get more clarity about this complication he is describing. A diagram might help things:

THE GENERAL PLANETARY HYPOTHESIS.

E = us, earth, the center of the ecliptic (a great circle on the celestial sphere)
D = center of the deferent for a planetary same-direction epicycle
Q = center of uniform motion for C, center of the epicycle [i.e. QC sweeps out equal angles in equal times around Q, not around D, although C always rides on the circle around center D]
QD = QE
(I have exaggerated the eccentricity in the figure.)



There is something eerily reminiscent of an ellipse in this diagram! In an elliptical orbit for a planet there will be two foci and a geometric center right between them. But more on that when we come to Kepler.

C, the center of the epicycle, is called the MEAN PLANET.

The RATIOS of speeds and of lengths in the figure will differ from planet to planet, and Ptolemy will be determining these. MERCURY has special problems which Ptolemy addresses, but we will stick to the less complicated planets. We will look at one inner planet, Venus, as far as the question "What are the ratios?" is concerned; and we will look at one outer planet, Saturn, as far as the question "Where do stations occur?" is concerned.

In the case of an INNER PLANET, where QC moves around Q with the speed of the mean sun, Ptolemy will assume that the line from E (Earth) out to the mean sun is always parallel to QC. (To get the proportions right, we would have to shrink the eccentricity quite a bit, and maybe grow the epicycle; in the case of Venus, the line joining Earth to the mean sun will always pass through the epicycle, almost through its center; and the apparent sun very nearly appears in line with the center of the epicycle!)

In the case of an OUTER PLANET, Ptolemy assumes that the line joining C to the planet (i.e. the epicyclic radius drawn to the star) is always parallel to the line joining E to the mean sun (i.e. from us to the mean sun).

In either case, if the lines were EVER parallel, they would always be parallel. But that they were parallel at any time in the past (or will be at some time in the future) is not known, or not well known, by naked-eye astronomical observation. But it certainly keeps things simple.

REMARK ON THE EQUANT.

To get a sense of how the equant works, you could make a circular track for a marble in a piece of plywood, and rotate an arm uniformly around some point *other* than the geometric center of the circle (but inside the circle). You will see the marble speed up and slow down in its circular groove.

This mechanism is a bit of a shift away from the ancient astronomical ideal, from the "astronomer's axiom" that the heavenly bodies move uniformly on perfect circles. Until we introduced this idea of an equant point, we had stars moving on perfect circles with uniform speed around the centers of those circles. That had a nice simplicity to it. But now we are divorcing the circular path that the planet moves on from the center around which it sweeps out equal angles in equal times! This will later scandalize Copernicus, who refuses to accept equants. Kepler (still later) loves equants, at least at first. But they will eventually get replaced by "the empty focus," in one way, and by the "full focus," the sun, in another way (with uniform area-velocity, as we shall see).

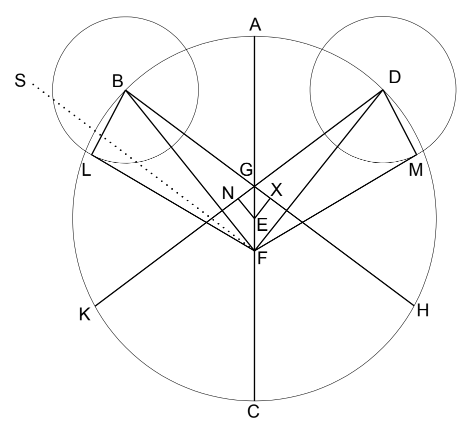
The need to posit an equant point will become clearer later. For now, Ptolemy is just preparing us for the idea, and I will generally include one in the diagrams.

IGNORING LATITUDINAL DIFFERENCES.

Ptolemy will be ignoring differences in latitude, since they make so little difference in longitudes.

THE "MICKEY MOUSE" PROPOSITION.

In Book 9 Chapter 6 of the *Almagest*, Ptolemy presents an argument which is needed later to find Venus's apogee in Book 10. It is not of great interest in itself, and might be called an over-elaboration. It is really a matter of symmetry. The proposition is as follows.



Given: Venus's eccentric deferent, line of apsides AEC
G = center of uniform motion (equant point)
F = eye
arcAB = arcAD (or ∠AGB = ∠AGD)
FL & FM are tangents to the epicycle at positions B and D

Prove: ∠GBF = ∠GDF
(angles of zodiacal anomalistic difference)
∠BFL = ∠DFM
(greatest elongations from the mean *planet*)

Draw: EN perpendicular to DGK
EX perpendicular to BGH

Now	$\angle XGE = \angle NGE$	[bc AGB = AGD, bc arcAB = arcAD]
and	$\angle GXE = \angle GNE$	[both right]
and	EG is common	
so	EN = EX	[$\triangle GXE = \triangle GNE$]
so	KD = BH	[equidistant from center]
so	ND = XB	[halves of KD & BH]
minus	NG = XG	[equal bc $\triangle GXE = \triangle GNE$]
so	GD = GB	[remainders]
but	$\angle DGF = \angle BGF$	[supplements to AGD & AGB]
and	GF is common	
so	BF = DF	[since $\triangle BGF = \triangle DGF$]
so	$\angle GBF = \angle GDF$	
But	BL = DM	[radius of epicycle]
and	$\angle BLF = \angle DMF$	[both right]
and	BF = DF	[proved above]
so	$\triangle BLF = \triangle DMF$	[1.47]
so	$\angle BFL = \angle DFM$	

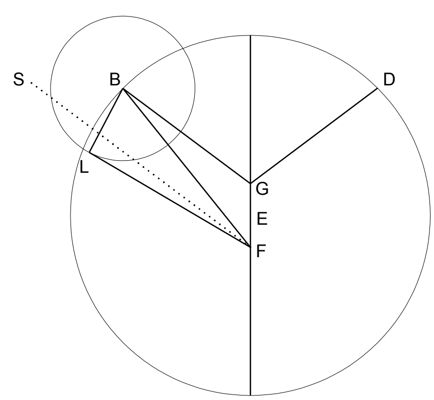
Q.E.D.

So, given equal angles on opposite sides of the line of apsides, we have proved that the greatest elongations (i.e. GEE on one side, GWE on the other) are equal.

QUESTIONS about the diagram:

Is BG parallel to LF?

No. BG is a line moving uniformly around G, and so it is always parallel to the line out to the mean sun S (for an inner planet).



Is L on the deferent?

No, not necessarily. But since F is not the center of the deferent, it is not impossible for L to be on the deferent.

PTOLEMY

DAY 23

VENUS'S LINE OF APSIDES;
THE NEED FOR THE EQUANT

In *Almagest* 10.1, Ptolemy proposes a demonstration concerning Venus's line of apsides. The goal is to find the orientation in the ecliptic of Venus's line of apsides, i.e. where its perigee and apogee occur in the ecliptic. The center of its deferent and the center of its uniform epicyclic motion ("equant point") will also lie along the same line.

Ptolemy thinks Theo the mathematician has good data, and relies on him. Anyway, the observations are of GEE (Greatest Eastern Elongation from the mean sun) and GWE (Greatest Western Elongation from the mean sun) for Venus, which he will use in order to find the line of apsides. From the converse of the "Mickey-Mouse" proposition back in Day 22, we see that when a GEE is equal to a GWE, these lie equal angles away from the line of apsides. So we just bisect, and that gives us the line of apsides.

(Note: due to conflicting influences of angle and distance, it might be possible to get several different GEEs at different times of year that are all the same, with unequal ones in between, and likewise for GWEs. But even if this is true, they would still have to be symmetrical, e.g. the 3 identical GEEs on one side of the line of apsides would occur at equal angles from it as 3 identical GWEs on the opposite side. So we can still just bisect, as long as we get enough data and see the symmetry.)

THE OBSERVATIONS.

Theo gives us a GEE for 16 Hadrian, Pharmouthi 21-22, when Venus was 1.5° within the Bull. At that time the mean sun was 14.25° within the Fishes. So the GEE at that time was 47.25°.

Ptolemy observed a GWE for 14 Hadrian, Thoth 11-12, when Venus was 18.5° within the Twins, and the mean sun was 5.75° within the Lion. So the GWE at that time was 47.25°.

Ptolemy notes that the members in this particular pair of GEE and GWE are equal, and hence symmetrically distant from the line of apsides for Venus. We bisect the angle between the locations of the mean sun during these elongations, i.e. 14.25° within the Fishes, and 5.75° within the Lion, and that gives us a line about 25° within the Bull and 25° within the Scorpion. There's our line of apsides.

Ptolemy then repeats the procedure, using another pair of GEE and GWE that are equal, just to confirm his results, and he does indeed get the same result.

In *Almagest* 10.2, Ptolemy determines:

1. Which end of its line of apsides Venus's apogee is on
2. The size of Venus's epicycle relative to its line of apsides

This is fairly easy. The epicycle will appear larger when it is at the perigee-end of the line of apsides, and smaller at the apogee-end. Hence the greatest elongations will be larger at the perigee-end. So we just have to find times when Venus is at its greatest elongation *and* the mean sun is at the opposite ends of the line of apsides.

So if the line of apsides passes through 25° into the Bull, and 25° into the Scorpion, how great are the greatest elongations which occur when the mean sun is in those places?

Ptolemy says that when the mean sun was (25 + 2/5)° within the Bull (close enough), Venus was (10 + 3/5)° in the Ram, hence west of the sun (hence a Morning Star), hence it had a GWE = (44 + 4/5)°.

Ptolemy then says that when the mean sun was 25.5° within the Scorpion (close enough), Venus was 12 + ½ + 1/3 ° within the Goat, hence east of the sun (hence an Evening Star), hence it had a GEE = 47.3°.

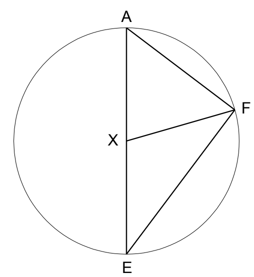
Obviously, then, the "Scorpion" end of our line of apsides is the perigeal end for Venus, and the "Bull" end is the apogeal end.

THE MAGNITUDE OF THE EPICYCLE
A.K.A. "THE BICYCLE PROP"

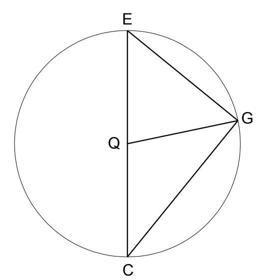
Here we are seeking the size of Venus's epicycle, i.e., how big it is relative to the deferent, and also its eccentricity.

Given: Circle ABC is Venus's eccentric deferent with center D
E is the center of the ecliptic
A is Venus's apogee
C is Venus's perigee
Epicycle at A, EF tangent
Epicycle at C, EG tangent

Find: Ratio of DE : AF : AD



Well, ∠AEF = greatest elongation at apogee
so ∠AEF = 44 + 4/5° (observed in Ch. 2, paragraph 2, end)
so arc AF = 89 + 36/60° (∠AXF = 2∠AEF, circle around △AEF)
so AF = 84^P 33' (by Table of Chords, if AE = 120)



Again, ∠CEG = greatest elongation at perigee
so ∠CEG = 47 + 1/3° (observed in Ch. 2, paragraph 3)
so arcCG = 94° 40' (∠CQG = 2∠CEG, circle around △CEG)
so CG = 88^P 13' (by Table of Chords, if CE = 120)

Thus CG : CE = 88^P 13' : 120
But CG = AF (being radii of the epicycle)
so AF : CE = 88^P 13' : 120
thus 84^P 33' : CE = 88^P 13' : 120
so CE = 115^P 1' (when CG = 84^P 33', and AE = 120)

Now AC = AE + EC
so AC = 120^P + 115^P 1' = 235^P 1' (where AF = 84^P 33' and AE = 120)
so AD = ½ AC = 117^P 30'
so DE = AE – AD = 120^P – 117^P 30' = 2^P 29' (where AE = 120^P)

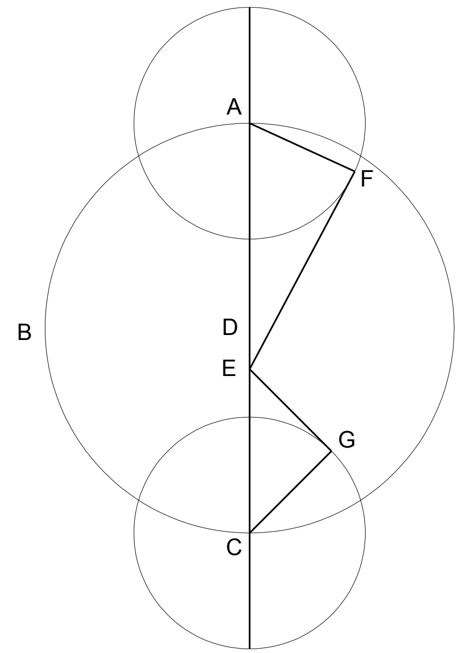
Thus DE = 2^P 29'

and	AF = 84^{P} 33’
and	AD = 117^{P} 30’
where	AE = 120^{P}

Hence	DE = 1 ¼
and	AF = 43 10’
and	AD = 60

by proportional translation, where AC = 120.

Q.E.I.



THE BRIGHTNESS PROBLEM

Is Ptolemy's hypothesis for Venus refuted because Venus's brightness would have to vary so much as not to match the appearances? This hypothesis will come up again in a certain preface to Copernicus's book (a preface he did not write) because it puts Venus so much closer to us at perigee than at apogee. Consider the numbers (rounding a bit for simplicity):

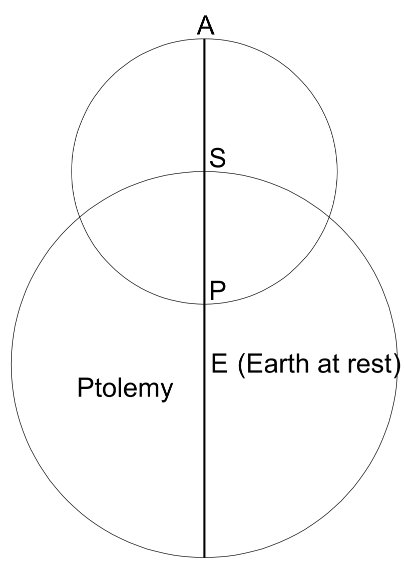
DE = 1
AF = 43
AD = 60

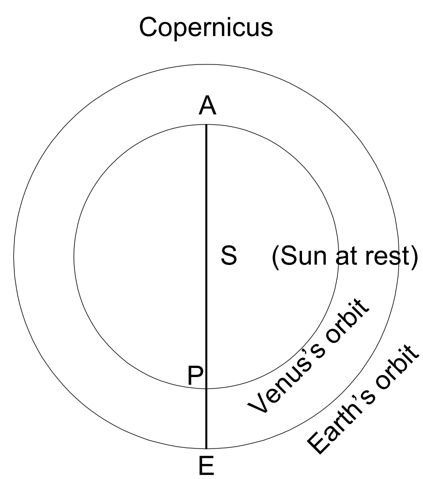
Apogeal distance = ED + DA + AF = 1 + 60 + 43 = 104
Perigeal distance = EC – CG = DC – DE – AF = AD – DE – AF = 60 – 1 – 43 = 16

So according to Ptolemy, Venus is about 10 times further from us at apogee than it is at perigee! But then it should appear remarkably brighter at perigee than at apogee, which is not the case. So is he refuted?

No, not for that reason. First of all, the case is just as bad for Copernicus. In each of the two accompanying diagrams (one for Ptolemy, the other for Copernicus),

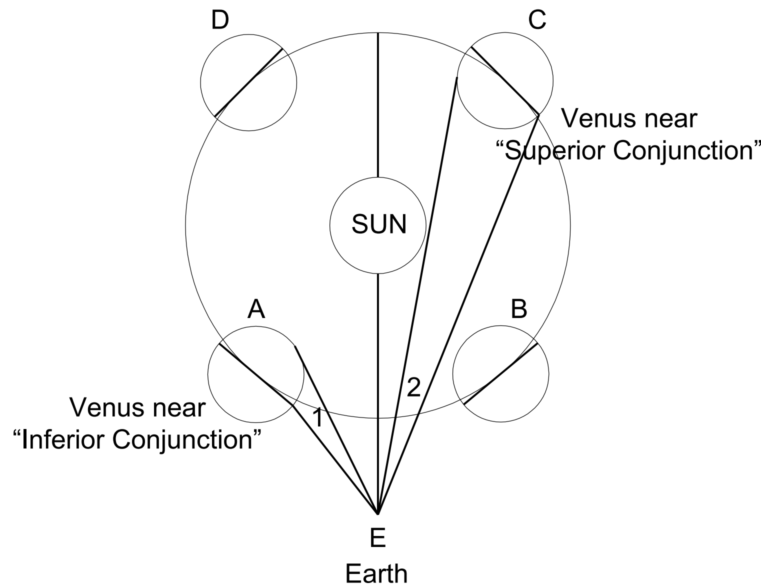
AS = 43 = SP
SE = 60
PE = 17
AE = 103





Second, when Venus is near perigee at A, much less of its lit-up face is visible to us, whereas near apogee at C, almost all its lit-up face is toward us, thus angles 1 & 2 will not be too unequal.

(Note: This reply assumes that Venus is *reflective only*, and not self-luminous! That is not something which Ptolemy knew, but perhaps he should have begun to suspect at this point.)

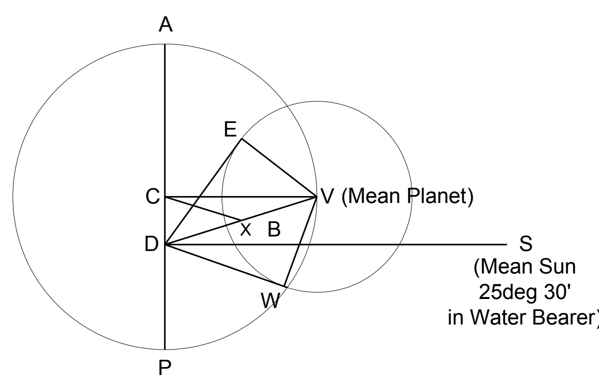


SEEING THE NEED FOR AN EQUANT.

In Day 24 we will be determining some things about Venus's equant. But before doing that, we need to see the need for adding in an equant in the first place. To see that, we need to answer two questions:

> [A] What are the two failures of the simple "eccentric deferent" hypothesis for Venus (which was determined in its proportions just above, following *Almagest* 10.2)?
>
> [B] What adjustment would make the hypothesis produce the appearances, and how?

[A] In the accompanying diagram, let the line DS from us to the mean sun be at right angles to AP, the line of apsides for Venus (namely when the mean sun is 25° 30' into the Water Bearer). According to our simple hypothesis of an eccentric deferent:



GEE = ∠SDE = ∠EDV + B

GWE = ∠SDW = ∠WDV – B

so GEE – GWE = (∠EDV + B) – (∠WDV – B)

so GEE – GWE = 2B = 2∠CVD = ∠CXD = arcCD

Now, by *Almagest* 10.2:

	CD = 1^P 15'
and	AP = 120^P
and	CV = 60^P

so $$DV = \sqrt{CD^2 + CV^2} = \sqrt{\left(1^P\ 15'\right)^2 + \left(60^P\right)^2} = 60^P$$

(i.e. DV is practically equal to CV)

so	CD = 1^P 15'
and	DV = 60^P
hence	CD = 2^P 30'
where	DV = 120^P

so arcCD = 2° 23' (for a circle around △CDV)
so GEE – GWE = 2° 23'
when the line from us out to the mean sun is perpendicular to AP, the line of apsides for Venus.

BUT IN FACT, when the mean sun is 25° 30' within the water bearer,

GEE – GWE = 4° 45'

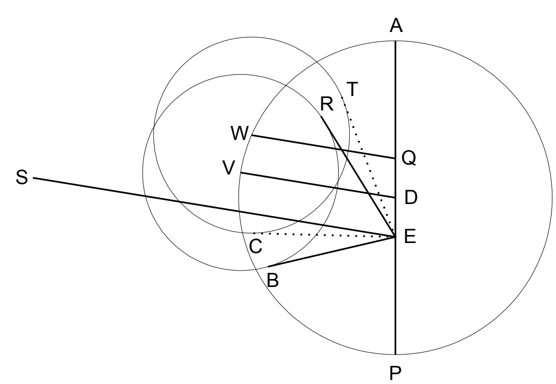
That's our first "uh-oh."
Our second "uh-oh" is that ∠SDE + ∠SDW is greater than what is actually observed, i.e. the sum of elongations for Venus which we can calculate based on our simple hypothesis (for when the mean sun is along DS perpendicular to AP) does not match the observed sum there.

So we can sum up our two "uh-ohs" this way:

(1) The *calculated* (GEE – GWE) < the *observed* (GEE – GWE)
(2) The *calculated* (GEE + GWE) > the *observed* (GEE + GWE)

[B] How must we adjust our hypothesis to match the appearances?

In the accompanying figure, ES is the line from us out to the mean sun, which sweeps around us uniformly. DV is the line parallel to this from the center of Venus's eccentric deferent, and hence DV sweeps around D uniformly with the speed of the mean sun.



Now, if we assume that the epicycle falls behind the regular motion around D (i.e. *behind* V, say at W), and then gradually catches up (throughout the motion from A to P), and then gets ahead of the regular motion around D (i.e. W gets *ahead* of V), while slowing down again (throughout the motion from P to A), it will follow that ...

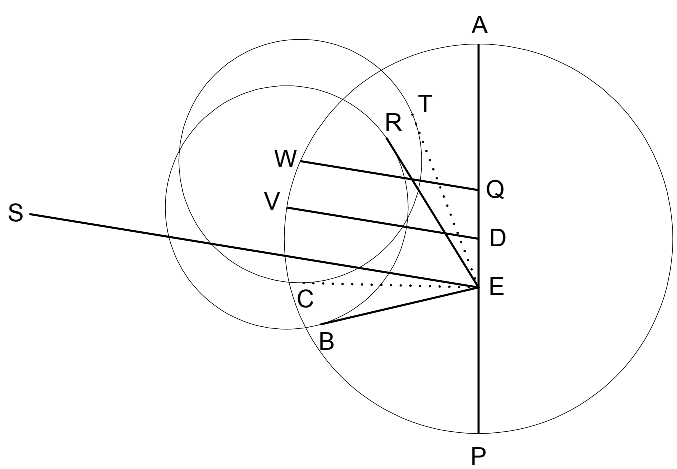
(2) The SUM of the elongations for V will be greater than those actually observed for W (and this is one appearance we need to produce), and

(1) The DIFFERENCE of the elongations for V will be less than those actually observed for W (the other appearance).

To see this, just let W be (for now) any point behind V along the motion from A to P. Then it is a matter of simple geometry to see that ∠BER > ∠CET, i.e. the calculated sum of elongations will be greater, since the "hypothetical" epicycle is closer to us when at V. Hence fact (2) is produced.

Again, since	∠BES > ∠CES
and	∠RES < ∠TES
thus	(∠RES – ∠BES) < (∠TES – ∠CES)

(since the lesser with more taken away is less than the greater with less taken away) and hence fact (1) is produced, i.e. the calculated difference of the elongations is less than the observed difference.



GETTING IT MORE EXACTLY

So far we have only said "we get things greater and less in the right order if W is behind and then ahead of V."

But in order to get things greater and less *in just the right amounts*, we need to specify how far behind W will be at certain points, and how far ahead at others.

Now the center of the epicycle for Venus has to move with the speed of the mean sun (since it coincides with the mean sun at apogee and perigee every time, so if it has a uniform speed it must be that of the mean sun). So if we draw WQ parallel to ES, that will hit the line of apsides at the point around which Venus's epicycle travels with that speed.

But how big is DQ compared to DE? Well, that's what *Almagest* 10.3 is all about ...

PTOLEMY

DAY 24

FINDING VENUS'S EQUANT CENTER; BASICS FOR THE OUTER PLANETS; THE MOVING ECCENTRIC EQUIVALENCE

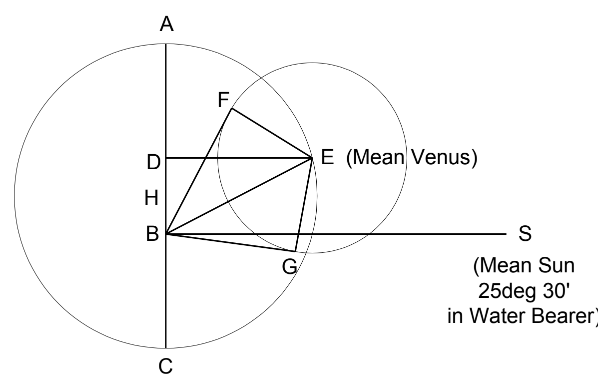
FINDING VENUS'S EQUANT CENTER

Ptolemy opens *Almagest* 10.3 with the line "Since it is not clear whether the regular movement of the epicycle is effected about point D . . ." After forewarning us about the equant, he will now establish the location of the equant's center. We find it by a kind of analysis: "Let it be required to find the center around which we say the epicycle's regular movement is effected. Then let it be [called] the point D," and now let's see exactly how far away it is from B (our eye) in comparison to the distance BH (where H is the geometric center of the deferent).

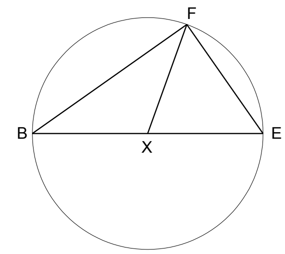
Given: B is us, the center of the ecliptic
D is the center of uniform motion for Venus's epicycle
BS is the line from us to the mean sun 25.5° into the Water Bearer, i.e. the point at which it "appears" 90° from Venus's apogee & perigee.

Find: Ratio of BD to BH

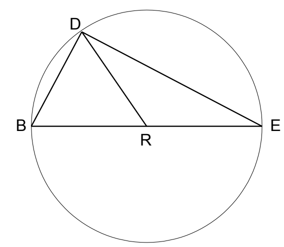
Draw DE parallel to BS.
Make the epicycle about E with the radius of the right proportion, as determined in 10.2. Draw tangents from the eye to the epicycle at that location, BF & BG.



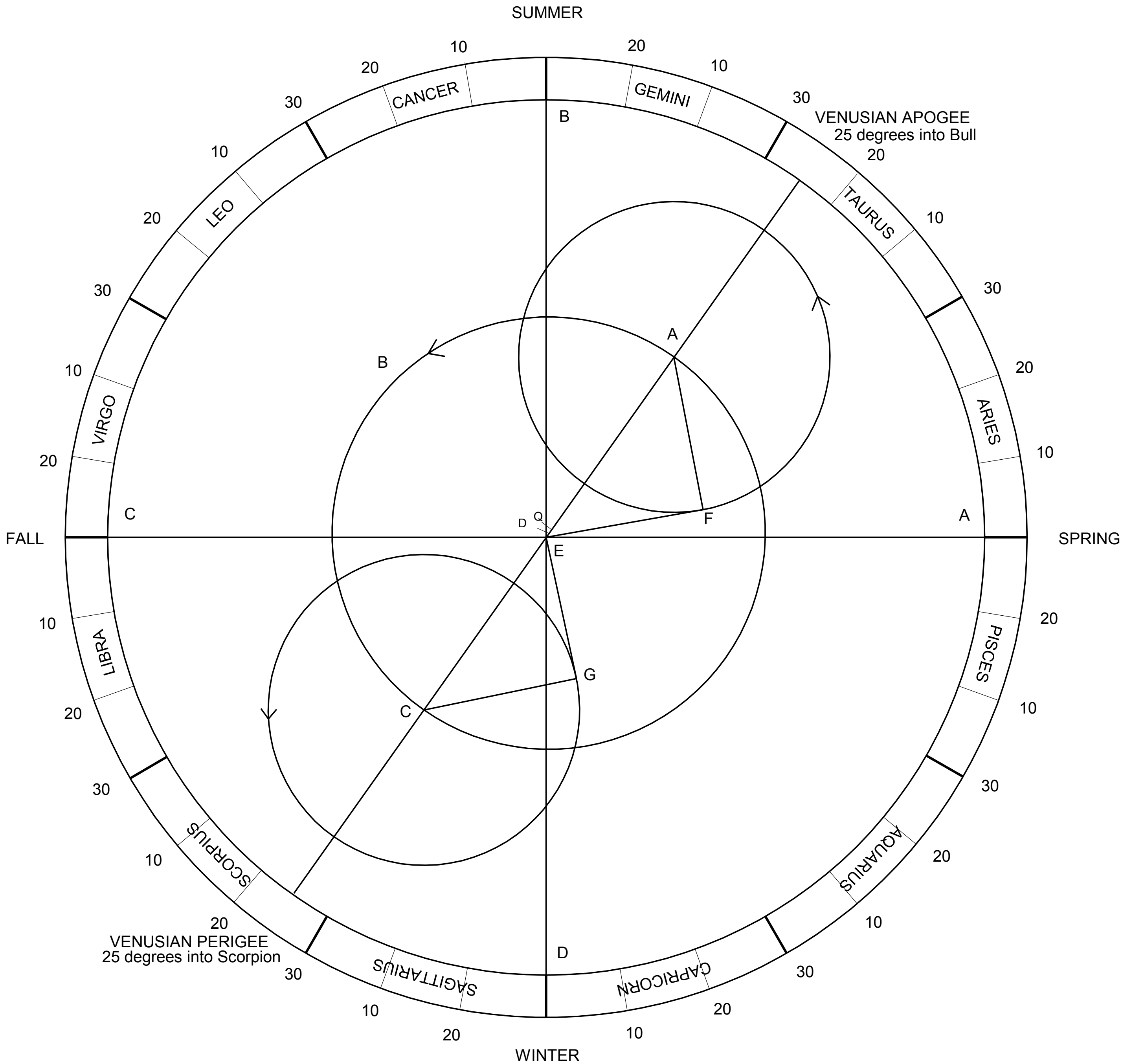
Now	∠FBS = GEE = 48 1/3° observed (evening star)
and	∠SBG = GWE = 43 7/12° observed (morning star)
so	∠FBG = 91° 55’
but	∠FBE = ½ FBG
and	∠FBE = ½ FXE
so	∠FBG = ∠FXE = 91° 55’ = arcEF
so	EF = 86^P 16’ when BE = 120^P
so if	EF = 43^P 10’ [as derived in Day 23]
then	BE = 60^P 3’



Also	(GEE – GWE) = 4° 45’ = 2∠BED
but	2∠BED = ∠BRD = arcBD
so	arcBD = 4° 45’
thus	BD = 4^P 59’ when BE = 120^P
or	BD = 2½^P when BE = 60^P 3’ and EF = 43^P 10’
But	BH = 1¼^P
so	BH = ½ BD.



Q.E.I.



HYPOTHESIS FOR VENUS (With its parts drawn to scale)

E = center of ecliptic (us) AD = 60
D = center of Venus's deferent AF = 43 1/6
Q = center of Venus's equant circle DE = 1 ¼

In this chapter, Ptolemy begins to introduce some fundamental notions about the motions of the outer planets. He supplies two demonstrations:

[1] *Whenever an outer planet & mean sun are in opposition or conjunction, our line of sight to the planet passes through the center of its epicycle.*

[2] *The line from the center of an outer planet's epicycle to the planet itself is parallel to the line from us to the mean sun.*

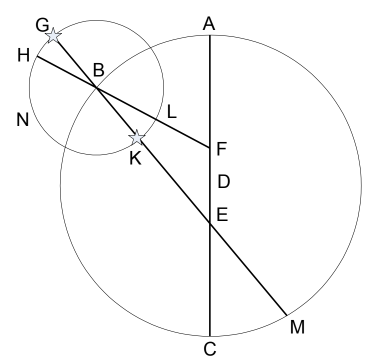
We will argue for these statements momentarily. But first, a few basics about outer planets.

• All 3 outer planets have equants.

• We used greatest elongations to know when Venus is at tangency on its epicycle, and used this information to find its eccentricity and the ratio of its epicycle to its deferent and the location of its apogee. Since outer planets have no greatest elongations from the sun, we need another method to gain insight into these planets. We will not cover this sort of thing in this course, mainly because it is more complex than is worthwhile for the purposes of this course (for interested readers, demonstration of Saturn's eccentricity & apogee is in *Almagest* 11.5, and demonstration of Saturn's size-of-epicycle is in 11.6).

• For an outer planet, we need to be able to find the center of the epicycle (as we did with Venus by taking sums of elongations that occur for given positions of the mean sun; bisect the sum of the elongations for that place in the zodiac, and you are bisecting the angle between the tangents to the epicycle, and shooting straight for the epicycle's center. And that is where the visible sun is, by the way, or pretty nearly). We will do this by demonstrating the 2 theorems of Ch. 6, enunciated above.

[1] *Whenever an outer planet & mean sun are in opposition or conjunction, our line of sight to the planet passes through the center of its epicycle.*



Given: Deferent of outer planet, center D.
F center of uniform motion of epicycle,
E center of ecliptic = us.
EB joining us to the center of the epicycle, B.
Case (1): Star is at G
Case (2): Star is at K

Prove: When star is at G, the mean sun "appears" at G.
When star is at K, the mean sun "appears" at M.

(1) When the star is at G,

then, since	S	=	L + A
thus	S	=	∠AFB + arcHKLG
i.e.	S	=	∠AFB + (360° – ∠HBG)

so	S = ∠AFB – ∠HBG + 360°	[you can just ignore this 360°]
i.e.	S = ∠AFB – ∠EBF	[∠HBG = ∠EBF]
so	S = ∠FEB	

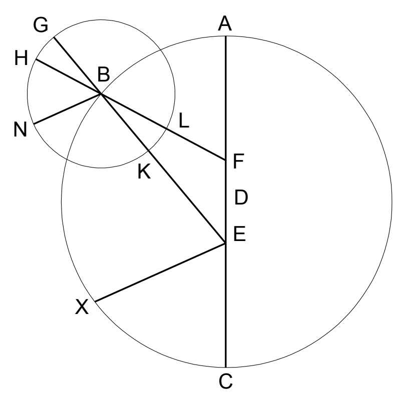
i.e. the mean solar movement is ∠FEB, and so when the planet is at G the mean sun "appears" along EB, i.e. the line from us through the center of the planet's epicycle. Here we have solar CONJUNCTION.

(2) When the star is at K,

then, since	S = L + A
thus	S = ∠AFB + arcHNK
i.e.	S = ∠AFB + (180° – ∠KBL)
so	S = ∠AFB – ∠KBL + 180°
i.e.	S = ∠AFB – ∠EBF + 180°
so	S = ∠FEB + 180°
so	S = arcABCM

i.e. the mean solar movement is arcABCM, and so when the planet is at K the mean sun "appears" along EBM, i.e. the line from us through the center of the planet's epicycle. Here we have solar OPPOSITION.

NOTE: G and K are the points of greatest and least passage respectively.



[2] *The line from the center of an outer planet's epicycle to the planet itself is parallel to the line from us to the mean sun.*

Given: Star is at N, epicycle at B, EX is parallel to BN
Prove: The mean sun "appears" along EX.

Since	S = L + A
thus	S = ∠AFB + ∠HBN
i.e.	S = ∠AFB + ∠GBN – ∠GBH

so $S = \angle AFB - \angle GBH + \angle GBN$
i.e. $S = \angle AFB - \angle EBF + \angle BEX$
so $S = \angle AEB + \angle BEX$
so $S = \angle AEX$
Therefore the mean sun appears along line EX.

Q.E.D.

Or, GIVEN that the mean sun is at X, we can prove that EX is parallel to BN:

Since $S = L + A$
thus $\angle AEX = \angle AFB + \angle HBN$
so $\angle AEX - \angle AEB = \angle AFB - \angle AEB + \angle HBN$
i.e. $\angle BEX = \angle EBF + \angle HBN$
or $\angle BEX = \angle GBH + \angle HBN$
so $\angle BEX = \angle GBN$
Therefore EX is parallel to BN.

Q.E.D.

In order for these proofs to work, Ptolemy must assume that at some point in the past or future the sun, the mean sun, Earth, Mars (for example), and the center of Mars's epicycle were (or will be) in a straight line. This makes the motions more intelligible anyway, and it is similar to the assumption that there *are* periodic joint returns, i.e. exact ones, not just approximate ones (although we might find out what they are only approximately).

Notice, too, that he assumes the equant before he can find where the center of the epicycle is, and therefore before he can even find the eccentricity, as opposed to the procedure for Venus, where the eccentricity is found independently of the equant.

Also, it is not clear that the assumption of an equant is warranted here. It is not like with Venus, where we get the basic hypothesis first and see that it needs adjusting. Here we introduce it at the outset, perhaps by analogy with the inner planets. At any rate, it turns out to work.

MOVING ECCENTRIC EQUIVALENCE

In *Almagest* 12.1, Ptolemy says:

> If the anomaly relative to the sun is taken care of by the hypothesis of the eccentricity (this can be done only in the case of the three stars capable of any elongation from the sun), with the eccentric's centre carried eastward about the ecliptic's centre at a speed equal to the sun's and with the star moving on the eccentric about its centre westward at a speed equal to the anomalistic passage etc. . . .

He is saying that the planets can be accounted for not just by an eccentric deferent and an equant, but also by some other "hypothesis of eccentricity," namely a moving eccentric circle. Just as he accounted for the movement of the sun by an eccentric but also by an epicycle, so too he has two ways of modeling the planets, at least the outer ones.

There is an unfortunate thing going on in the Greek which makes it seem as though Ptolemy is saying the eccentric hypothesis will work only for the outer planets (this is his remark in the parentheses in the passage above). That is not what he is saying. The limitation he implies is supposed to modify what comes after it, so it should be understood more like this:

> If the anomaly [of any planet] relative to the sun is taken care of by the hypothesis of the eccentricity (*and this additional thing I'm about to mention can be done only in the case of the three stars capable of any elongation from the sun*), with the eccentric's centre carried eastward about the ecliptic's centre at a speed equal to the sun's and with the star moving on the eccentric about its centre westward at a speed equal to the anomalistic passage etc. ...

So the only thing restricted to the 3 outer planets is "the equality of the speeds of the moving eccentric and of the sun."

Anyway, let's take a moment to see what the equivalent hypothesis of the "moving eccentric" is.

MOVING ECCENTRIC EQUIVALENCE

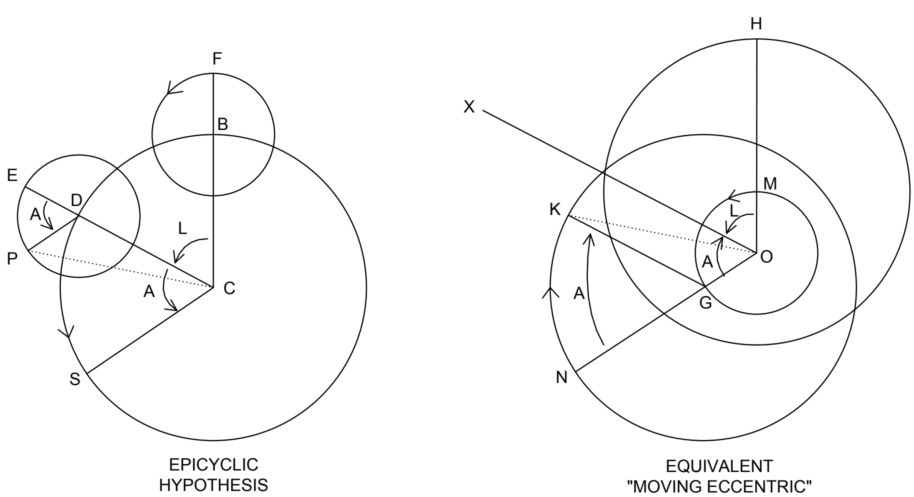
Given: An outer planet on a same-direction epicycle with deferent of center C, where we are (we are ignoring the eccentricity of the deferent to keep things simple); in time T the epicycle moved ∠BCD = L, the star moved ∠EDP = A, and the mean sun moved ∠BCS = S [remember CS is parallel to DP for the outer planets].

The same outer planet, on eccentric of center M which is moving about O where we are; in time T the eccentric moved ∠MOG = L + A = S, the star moved *westward* on the eccentric through ∠NGK = A,

and OG = DP,

and GN = CD.

Prove: ∠BCP = ∠HOK (apparent motions)



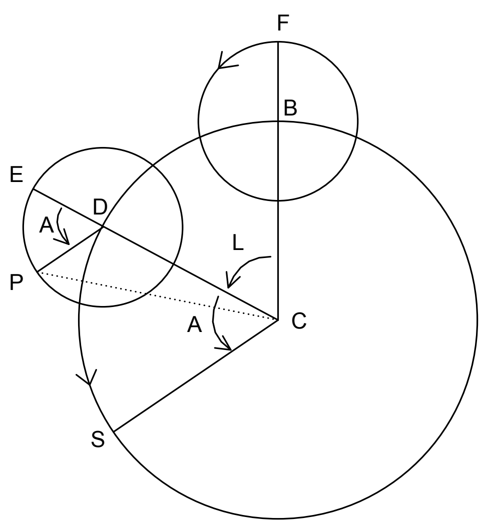
Well,	$\angle KGN = A$,	$\angle EDP = A$	
so	$\angle KGO$ suppl.A,	$\angle PDC$ suppl.A	
so	$\angle KGO = \angle PDC$		
but	$OG = DP$		
and	$GK = CD$		[since GK = GN, and GN = CD]
so	$\triangle KGO = \triangle PDC$		
so	$\angle KOG = \angle CPD$		

Again	$\angle MOG = L + A = S$,	$\angle BCS = S$	
so	$\angle MOG = \angle BCS$		
minus	$\angle KOG = \angle CPD$		[proved above]
so	$\angle MOK = \angle BCS - \angle CPD$		[remainders]
i.e.	$\angle HOK = \angle BCS - \angle PCS$		[CS parallel to DP]
so	$\angle HOK = \angle BCP$		

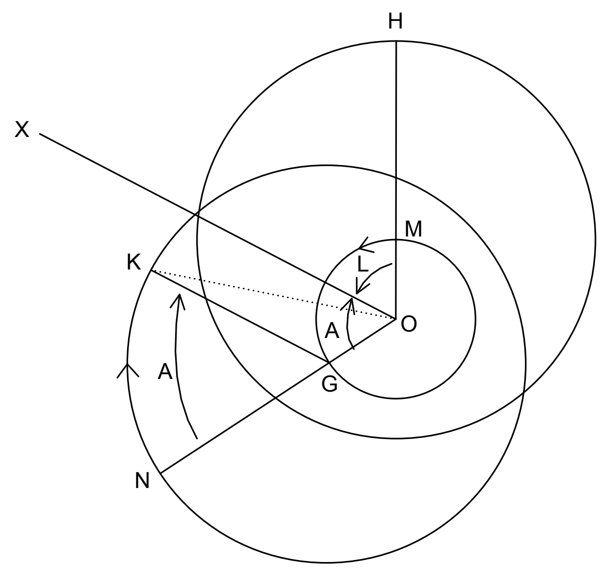
Q.E.D.

In general, all we need is:

PD : DC = OG : GK.



EPICYCLIC HYPOTHESIS



EQUIVALENT "MOVING ECCENTRIC"

NOTE: In the "moving eccentric" hypothesis, the mean sun is always on the line of apsides (of the eccentric), i.e. along OH at first (it is assumed to be up there), and then along ON.

So $\angle HON = S = L + A$.

But the meanings of "L" and "A" are different, since we no longer have an epicycle. "L" and "A" still mean the same in terms of appearances, i.e. "L" is the mean motion around us, and "A" is the heliacal anomaly. But now we are saying the cause of "A" is westward motion along the moving eccentric, while the cause of "L" is the fact that the moving eccentric moves with the speed of the mean sun, which is L + A.

So when we say "S = L + A" now, for the moving eccentric, we mean the appearances, while L has no separate "incarnation" in this hypothesis, like it does in the case of the epicycle (where it is the movement of the center of the epicycle around the equant's center).

PTOLEMY

DAY 25

We are now moving on to material which Ptolemy covers in Book 12 of the *Almagest*: stations. We want to be able to predict stations, and to say exactly where the planet is on its epicycle when a station occurs.

(1) STATIONS DO NOT OCCUR AT TANGENCY.
It is easy to think that the stations will occur when the line from our eye to the planet is tangent to the epicycle. But that is not the case! It is not hard to see why that cannot be the case: at tangency, motion on the epicycle is temporarily nullified in the appearances, leaving us with the speed of the mean planet, i.e. the planet will appear to move with the apparent motion of the epicycle itself. That is not nothing. We aren't looking for where one speed contributes nothing to the appearances and the other is left over. We need to find a place where the two speeds *cancel* each other out in the appearances. Where precisely does that happen, and why?

(2) RECALLING THE APPEARANCES.

Let's look to some old raw data about Saturn's retrograde motions, of the sort that Ptolemy would have had available to him.

[a] Saturn first began retrograde motion on June 26, 1993, after STATION from June 6 to 16, 1993. [Call this SE, or Eastern Station]

[b] Saturn next resumed its eastward motion on Nov 13, 1993, after STATION from Oct 24 to Nov 3, 1993. [Call this SW, or Western Station]

[c] Saturn next resumed retrograde motion on July 1, 1994, after STATION on June 21, 1994 [Call this Se].

Thus the time from SE to Se (from station to the next *corresponding* station) = 1 year + 15 days, i.e. it happens about once a year (around solar opposition), but takes a bit longer because Saturn moves slowly eastward on the ecliptic, so it takes the sun a bit longer than a year to catch up with it again (or become opposite to it again).

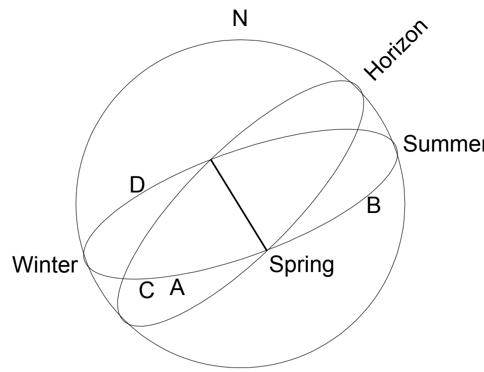
[d] The time from SE to SW = June 6 1993 to Oct 24 1993 = 140 days

The time from SW to Se = Oct 24 1993 to June 21 1994 = 240 days

[e] At SE, longitude of Saturn = 330.3° east of spring equinox = A
and longitude of mean sun = 435.3° east of spring equinox,
but minus the full circle (360°) = 75.3° east of spring equinox = B

Thus Saturn rises *before* the sun.

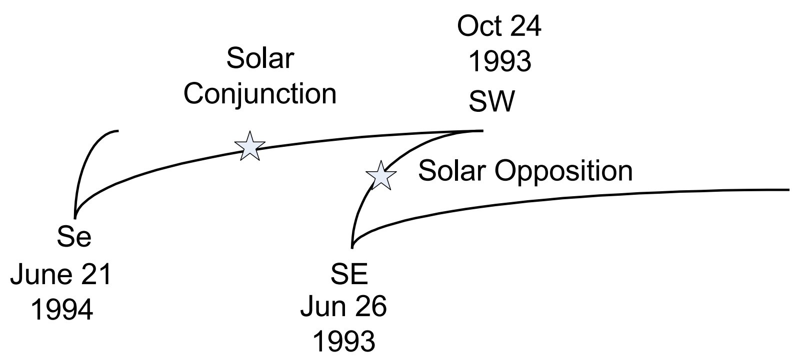
Midway between SE & SW is around 70 days after SE, i.e. Aug 15 1993, where the longitude of Saturn = 327.3°, longitude of mean sun = 504.3°, and therefore Saturn has an elongation of 177°, i.e. almost exactly 180° of elongation, i.e. near solar opposition. Therefore retrogradation is considered a "heliacal" anomaly, because it happens during *solar opposition*.



At SW, Saturn's longitude = 327.3° = C
mean sun's longitude = 573.3° = (minus 360°) 213.3° = D

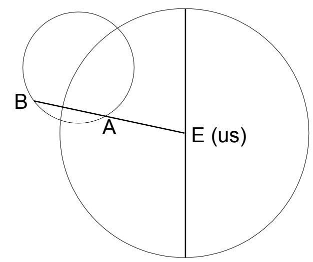
Thus Saturn rises *after* the sun.

Looking in the sky, facing south, one sees this:



Midway between SW and Se is about 120 days after SW,
i.e. Feb 21 1994, where
Longitude of Saturn = 332.9° east of spring equinox
Longitude of mean sun = 691° = (minus 360°) 331.7° east of spring equinox
So Saturn's elongation = 1.2°, i.e. solar *conjunction*.

(3) BACK TO THE STATION PROBLEM.
Ptolemy makes certain assertions about where stations occur, although he does not immediately prove them. (Note that by “star speed” he means the speed of the planet around the epicycle.) He says:

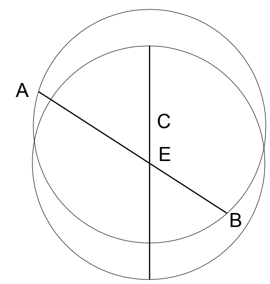


For the EPICYCLIC hypothesis, for any planet,

IF ½ AB : EA = epicycle speed : star speed

THEN point A divides progressions & regressions, which is to say that the star makes a station when at A.

(Motion on deferent = L, motion on epicycle = A)



For the moving ECCENTRIC hypothesis, for any planet,

IF ½ AB : EB = eccentric speed : star speed

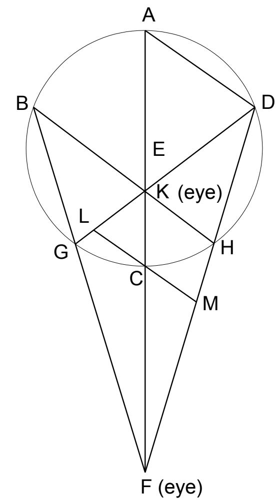
THEN point B divides progressions & regressions, which is to say that the star makes a station when at B.

(Motion of eccentric’s center C = L, motion of star *westward* along the eccentric = A).

But these facts are not demonstrated until we get to the finding of station-points in Day 26.

Before we do that, we need to do some preliminary geometry with Ptolemy.

THEOREMS PRELIMINARY
TO DEMONSTRATION OF STATION POINTS



DEMONSTRATION 1

NOTE: These theorems are not really necessary to demonstrating that station occurs here or there, but to demonstrating how the moving eccentric is EQUIVALENT to the epicyclic hypothesis. The mathematics is rather beautiful.

Given: Line from point F outside an epicycle (or eccentric) drawn through the center E
arcCG = arcCH
FGB & FHD joined through
hence GD & HB meet at one point K (on diameter CA)

Prove: AF : FC = AK : KC (so the diameter is cut "externally & internally" in the same ratio)

Join AD, DC (thus ∠ADC is right)
Draw LCM parallel to AD (thus ∠LCD is right).

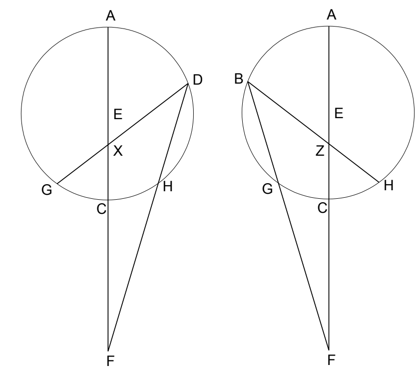
Now	∠CDG = ∠CDH	[they stand on equal arcs]
and	∠DCL = ∠DCM	[both right]
and	CD common	
so	△DCL = △DCM	
so	CL = CM	
thus	AD : CL = AD : CM	
but	AD : CM = AF : CF	[△AFD similar to △CFM]
and	AD : CL = AK : CK	[△AKD similar to △CKL]
so	AF : CF = AK : CK	

Q.E.D.

DISCUSSION:

(1) Getting back to astronomy for a moment:
for the EPICYCLIC, with the eye at F,
AF : CF = greatest distance : least = a : b;
for the ECCENTRIC, with the eye at K,
AK : CK = greatest distance : least = a : b.

(2) QUESTION: Is K (i.e. the point where BH & GD intersect) really on the diameter? This is fairly obvious from the symmetry of the figure. But we can also argue like this:



let GD cut the diameter at point X,
and BH cut the diameter at point Z, without committing ourselves to saying whether these are the same point or not. In the demonstration above, we used K only as a point on GD, and so we can still say, from that demonstration:

AX : XC = AF : FC

but AZ : ZC = AF : FC by the same argument on the other side.

hence X & Z are the same point.

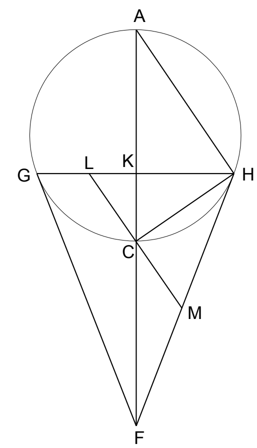
Hence GD and BH cut the diameter at the same point, and hence cut each other at the point where they cut the diameter. So K is indeed on the diameter.

(3) QUESTION: Does the point K stay the same for all secants?

Notice that we produced point K by a random secant FGB (and its symmetrical partner FHD), from the given point F. But AF : CF is fixed, and no matter where we draw FGB the proof will still result in the proportion AK : KC = AF : FC. Hence K will always be the same fixed point, regardless of the secants we start with. Neat!

(4) Notice, too: if we join the points of tangency for the two tangents drawn from F, that line will pass through K as well:

Let FG and FH be the two tangents, FCA the diameter, and join GKH. I say that once again AK : KC = AF : FC ...



For let LCM be drawn parallel to AH, and let CH be joined.

Now	∠CHK = ∠CAH	[KH is perpendicular to AC, ∠AHC is right]
but	∠CHM = ∠CAH	[Euc.3.32, tangent & angle in alt. segment]
thus	∠CHK = ∠CHM	
but	∠LCH = ∠HCM	[both 90°]
thus	△LCH = △HCM	[ASA]
so	CM = CL	
thus	AH : LC = AH : CM	[since CM = CL]
but	AH : LC = AK : KC	[AH parallel to LC]
and	AH : CM = AF : FC	[AH parallel to CM]
so	AK : KC = AF : FC	

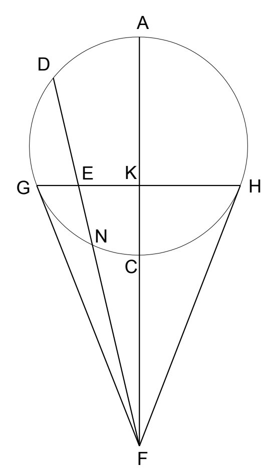
Q.E.D.

(5) HARMONIC MEAN. Note that FK is the harmonic mean between FC & FA.

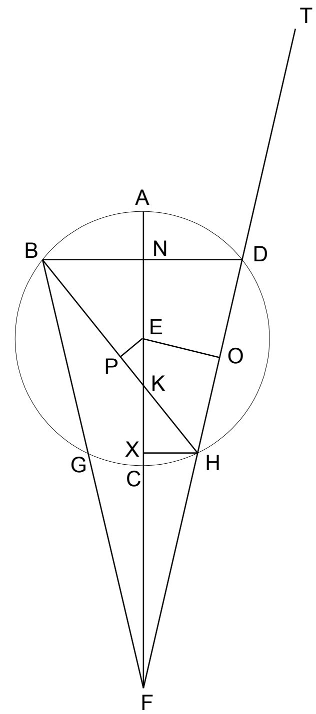
For	AK : KC = AF : FC	
so	AK : AF = KC : FC	[alternating]
so	AF – FK : AF = FK – FC : FC	[AK = AF – FK; KC = FK – FC]

and that is the definition of a harmonic mean, i.e. a thing (FK) whose difference from one extreme (AF) is to that extreme (AF) as its difference from the other extreme (FC) is to that extreme (FC).

(6) SMELLING THE FLOWERS. If we draw secants through from F, they will be cut internally and externally in the same ratio by GH, the straight line joining the points of tangency. (DF : FN = DE : EN) This beautiful thing follows from the theorem Ptolemy has just proved. No time or need to prove it here, but it is interesting, and very important to the study of conic sections by projection.



(7) NEXT PRELIMINARY.



Given: External point F, arcGC = arcCH
FGB & FHD drawn through
BKH joined
EP perpendicular to BKH
EO perpendicular to FHD

Prove: DF : FH = BK : HK
HO : FH = KP : HK

Join BND (obviously BND is perpendicular to AC).
Drop HX perpendicular to AC.

Now BN = DN [BND is perpendicular to diameter AC]
so DN : HX = BN : HX
but DN : HX = DF : FH [$\triangle$DNF similar to $\triangle$HXF]
and BN : HX = BK : HK [$\triangle$BNK similar to $\triangle$HXK]
so DF : FH = BK : HK

Thus DF + FH : FH = BK + HK : HK
so ½ (DF + FH) : FH = ½ (BH) : HK
i.e. FO : FH = HP : HK
so FO – FH : FH = HP – HK : HK
so HO : FH = KP : HK

Q.E.D.

QUESTION: How do we know that FO = ½ (DF + FH)?
Well, let DT = HF
so FO = ½ FT
so FO = ½ (DF + DT)
so FO = ½ (DF + FH)

(8) THE EQUIVALENCE OF STATION POINTS FOR THE 2 HYPOTHESES.

After the first demonstration, Ptolemy says:

> If therefore, in the epicyclic hypothesis, DF has been so drawn that HO : FH = epicycle's speed : star's speed, then, in the hypothesis of eccentricity, KP will have to HK the same ratio. And the reason for not having used here, for the stations, the **separated ratio** (that is, the ratio of KP to HK) but the **unseparated ratio** (that is, the ratio of HP to HK) is that the epicycle's speed has to the star's speed the ratio which the longitudinal passage alone has to the anomalistic; but the eccentric's speed has to the star's the ratio which the sun's mean passage (that is, the star's longitudinal and anomalistic passages combined) has to the anomalistic.

The "separated ratio" refers to KP : HK
while the "unseparated" refers to HP : HK.

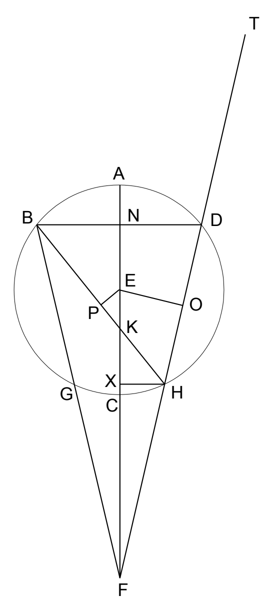
Ptolemy just showed that HO : FH = KP : HK. Now the ratio HO : FH (as we shall see) defines the EPICYCLIC station points. So one might think that therefore KP : HK defines the ECCENTRIC station points. But that ratio does not, and Ptolemy is here explaining why. For the EPICYCLIC, a station occurs when:

HO : FH = epicycle speed : star speed = L : A

For the ECCENTRIC a station occurs when:

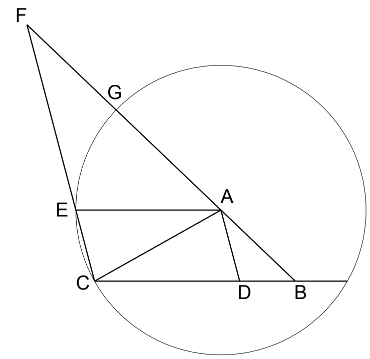
HP : HK = eccentric speed : star speed = L + A : A = KP + HK : HK

Because the eccentric's speed combines L + A, we get the eccentric's station by the "unseparated" ratio, HP : HK.



(9) ONE MORE LEMMA.

Ptolemy introduces another Lemma, saying that "**APOLLONIUS** first takes this little lemma . . .". This lemma *is* required for determining where station occurs.



Given: △ABC in which BC > AC
CD ≥ AC

Prove: CD : BD > ∠ABC : ∠BCA

Complete parallelogram ADCE.
Produce BA & CE to F.

Now CD ≥ AC
so AE ≥ AC [CD = AE]
So the circle about A, radius AE, passes through C or beyond it, as at H (in diagram below).

FIRST let it pass through C, so that AE = AC.

So △AEF > Sect.AEG
so △AEF : Sect.AEC > Sect.AEG : Sect.AEC
so △AEF : △AEC > Sect.AEG : Sect.AEC [Sect.AEC > △AEC]
But △AEF : △AEC = FE : EC
and <u>Sect.AEG : Sect.AEC = ∠EAF : ∠EAC</u>
so FE : EC > ∠EAF : ∠EAC [call this result "Q"]
but FE : EC = FA : AB = CD : BD
and <u>∠EAF : ∠EAC = ∠ABC : ∠BCA</u> [∠EAF = ∠ABC, ∠EAC = ∠BCA]
so CD : BD > ∠ABC : ∠BCA

NEXT let the circle about A, radius AE, pass beyond C at H, so that AE > AC. (2nd CASE)

Thus △AEF > Sect.AEG
so △AEF : Sect.AEH > Sect.AEG : Sect.AEH
but <u>∠EAF : ∠EAC = Sect.AEG : Sect.AEH</u>
so △AEF : Sect.AEH > ∠EAF : ∠EAC
but <u>Sect.AEH > △AEC</u>
so △AEF : △AEC > ∠EAF : ∠EAC
so FE : EC > ∠EAF : ∠EAC [△AEF : △AEC = FE : EC]
And now just continue the reasoning again from [Q] in the first case above.

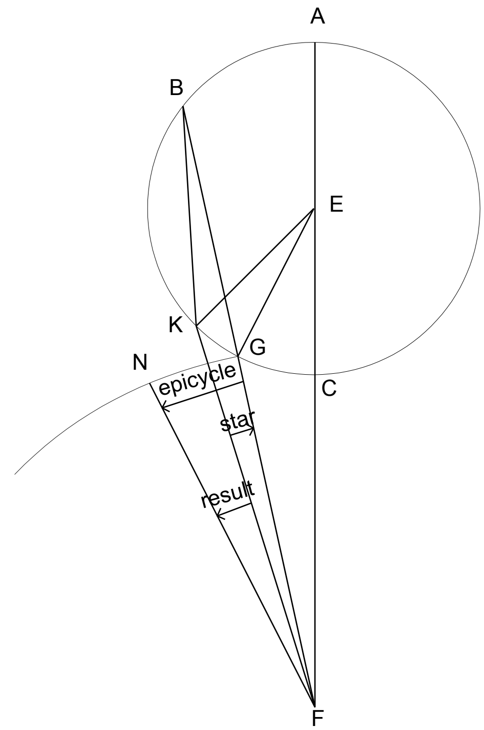
Q.E.D.

PTOLEMY

DAY 26

DETERMINATION OF STATIONS

EPICYCLIC PROGRESSION PROOF:



Given: F is our eye
CE : CF > epicycle speed : star speed
and hence we can draw secant FGB so that
½ BG : FG = epicycle speed : star speed
K is a random point on arcAG

Prove: When the star goes through arcKG, it is in PROGRESSION

Join FK through to L (only for regression part later).
Join BK, EK, EG.

Now, BG > BK [closer to center]
so BG : FG > ∠GFK : ∠GBK [Lemma, Day 25]
so ½ BG : FG > ∠GFK : 2∠GBK
i.e. epicycle speed : star speed > ∠GFK : ∠GEK

But if we give ourselves an angle GFN through which the epicycle moved in the same time that the star moved through angle GEK,
then epicycle speed : star speed = ∠GFN : ∠GEK
hence ∠GFN > ∠GFK. (To adjust the disproportion to a proportion.)

So, while the epicycle goes from G to N,
the star goes from K to G.

So to our eye, the star goes from K to N, which is PROGRESSION.

Q.E.D.

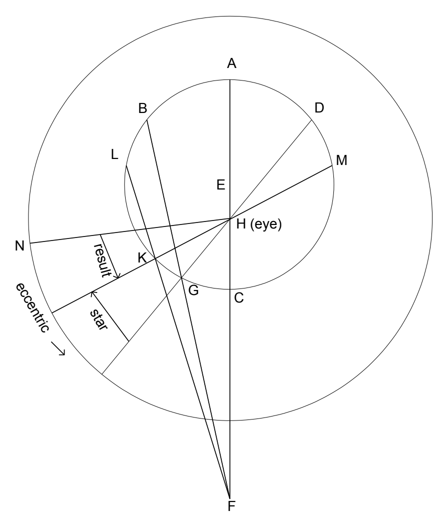
Incidentally, if 1 : 3 = ∠GFN : 60° [i.e. if the epicyclic speed is to the star speed as 1 to 3, and ∠GEK = 60°], then ∠GFN = 20°, and we can't construct this without conics. And if the epicycle speed is to the star speed in some irrational ratio, then maybe we cannot construct ∠GFN at all.

But that's okay, because ∠GFN is not a math construction, but the angle swept out by point G on the star's epicycle in a given time, i.e. it is a physical reality.

ECCENTRIC PROGRESSION PROOF:

Given: H is our eye (and arcAB = arcAD, GHD joined)
CE : CF > epicycle speed : star speed
½ BG : FG = epicyclic speed : star speed
but ½ BG : FG = ½ DG – HG : HG (by Lemma, Day 25)
so ½ DG – HG : HG = epicyclic speed : star speed = L : A
so ½ DG – HG + HG : HG = L + A : A
so ½ DG : HG = L + A : A
i.e. ½ DG : HG = eccentric speed : star speed
K is a random point on arcAG.

Prove: When the star goes through arc AG, it is in PROGRESSION.



Join FK through to L, KH through to M.
Since H is the harmonic point for secants from F (since arcAB = arcAD),
thus arc AL = arc AM
thus arc BL = arc DM.

Now BK < BG
so BG : FG > ∠GFK : ∠GBK [Lemma, Day 25]

so BG + FG : FG > ∠GFK + ∠GBK : ∠GBK
i.e. BF : FG > ∠BKL : ∠GBK [Euc. 1.32]
so DH : HG > ∠BKL : ∠GBK [BF:FG = DH:GH, Thm p.394]
or DH : HG > ∠DKM : ∠GBK [arcBL = arcDM]
so DH : HG > ∠DKM : ∠GDK [∠GBK, ∠GDK on same arc GK]
so DH + HG : HG > ∠DKM + ∠GDK : ∠GDK
i.e. DG : HG > ∠GHK : ∠GDK [Euc. 1.32]
so ½ DG : HG > ∠GHK : 2∠GDK
so ½ DG : HG > ∠GHK : ∠GEK [angle at center is double]

But, by our givens,

½ DG : HG = eccentric speed : star speed

and therefore

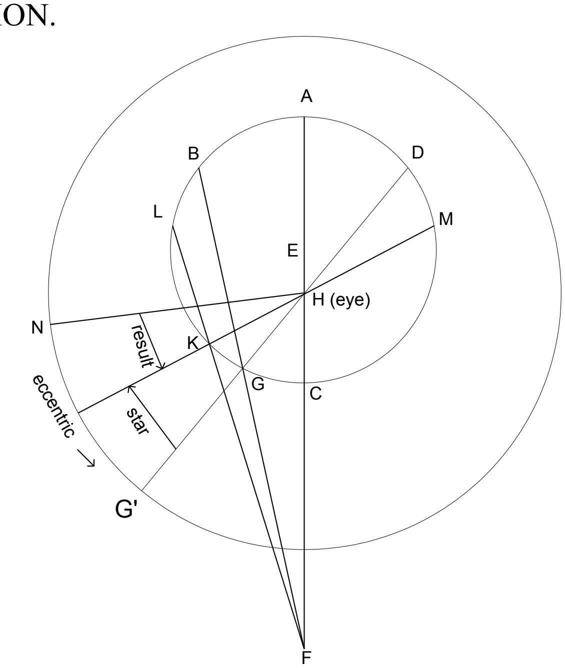
eccentric speed : star speed > ∠GHK : ∠GEK.

So if eccentric speed : star speed = ∠GHN : ∠GEK,

it follows ∠GHN > ∠GHK.

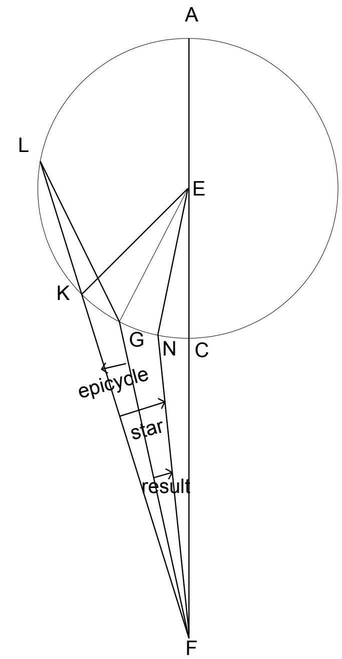
But the eye is at H. Therefore, while the star goes from G to K on the eccentric, in the same time the eccentric circle itself goes from N to G, and thus the star appears to go from N to K, which is PROGRESSION.

Q.E.D.



NOTE: The outer circle is drawn just to indicate the motion of the eccentric LBADM around point H, i.e. from N to G'.

EPICYCLIC & ECCENTRIC REGRESSION



EPICYCLIC REGRESSION PROOF:

Given: F is our eye
CE : CF > epicycle speed : star speed
and hence we can draw secant FKL so that
½ KL : FK = epicycle speed : star speed
G is a random point on arcKC

Prove: When the star goes through arcKG, it is in REGRESSION
i.e. it retrogrades through ∠GFN during the time it goes through arcKG

Well, FK > FG
so KL : FK < ∠GFK : ∠GLK [Lemma, Day 25]
so ½ KL : FK < ∠GFK : 2∠GLK
so ½ KL : FK < ∠GFK : ∠GEK
thus epic.spd : star spd. < ∠GFK : ∠GEK [½KL : FK is as speeds]
So if we replace ∠GEK with ∠KEN so as to make the ratios equal,

then ∠KEN > ∠GEK
and epic.spd : star spd. = ∠GFK : ∠KEN

So in the same time that the epicycle progresses through angle GFK, the star will sweep out angle KEN around the center of the epicycle, so that the star's net motion will be from G to N, i.e. angle GFN to our eye, which is REGRESSION. Hence, too, if we replace ∠GFK with ∠X to make the ratios equal (Ep. : Star = X : GEK), then X < GFK, and so in time star goes from K to G, epicycle goes less than ∠GFK, hence the star is regressing.

Q.E.D.

Q: Why give that CE : CF > ep. : star ? (Otherwise we can't draw the secant.)
Q: What if star spd. = zero? (The planet is always in progression.)
Q: What if ep. spd. = zero? ("Stations" are at the tangent points!)
Q: What if CE : CF = ep. : star ? (One station at C.)
Q: What if CE : CF < ep. : star ? (No stations; all progression.)

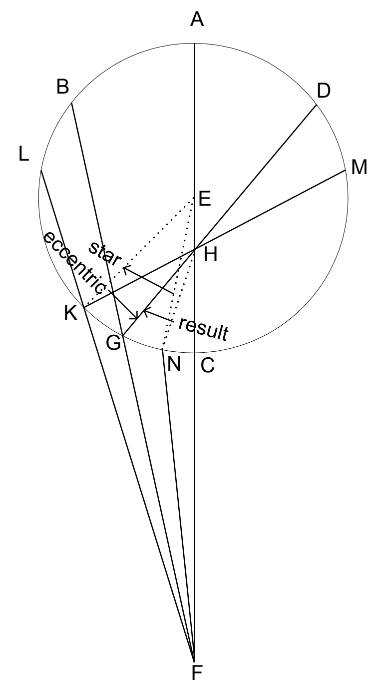
Exercise: In a diagram, correlate (with letters) the planet's zig-zag (or loop) pattern in the sky with the places the star is on the epicycle.

ECCENTRIC REGRESSION PROOF:

Given: H is our eye
$\frac{1}{2}$ KM : HK = eccentric spd. : star spd.
(the "unseparated ratio" corresponding to $\frac{1}{2}$ KL : FK for the epicyclic)
G is a random point on arcKC

Prove: When the star goes through arc KG, it is in REGRESSION.

Well,	$KL : FK < \angle GFK : \angle GLK$	[as before, for epicycle, above]
so	$KL + FK : FK < \angle GFK + \angle GLK : \angle GLK$	
so	$FL : FK < \angle BGL : \angle GLK$	[Euc.1.32]
so	$MH : HK < \angle BGL : \angle GLK$	[p.394]
so	$MH + HK : HK < \angle BGL + \angle GLK : \angle GLK$	
so	$KM : HK < \angle DGM + \angle KMG : \angle GLK$	
i.e.	$KM : HK < \angle KHG : \angle GLK$	[Euc.1.32, $\triangle$GHM]
so	$\frac{1}{2} KM : HK < \angle KHG : 2\angle GLK$	
so	ecc.spd : star spd $< \angle KHG : \angle GEK$	



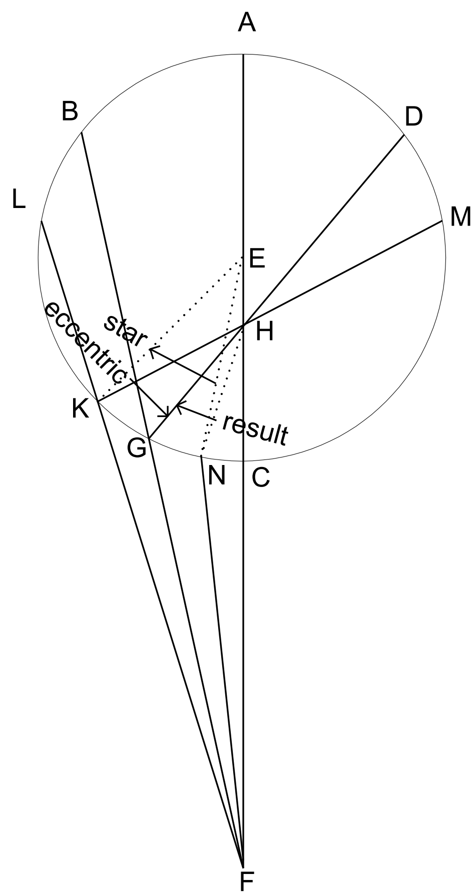
So if we replace $\angle GEK$ with $\angle KEN$ so as to make the ratios equal

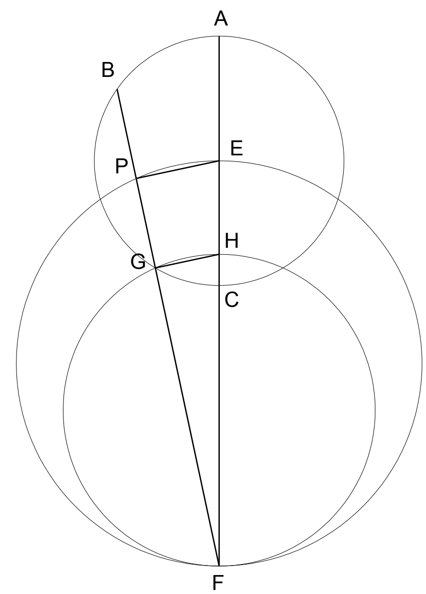
then	$\angle KEN > \angle GEK$:
and	ecc.spd : star spd. $= \angle KHG : \angle KEN$

So in the same time that the eccentric progresses through angle KHG, the star will sweep out angle NEK around the center of the eccentric, so that the star's net motion will be from N to G, i.e. angle NHG to our eye, which is REGRESSION.

Q.E.D.

NOTE: If CE : CF is not greater than e : s, then either CE : CF = e : s, in which case you get one station at C, or CE : CF < e : s, and we get no station at all, because, given the foregoing proofs, progression will always produce the greater angle, and therefore we get nothing but progression.





CONSTRUCTION OF THE STATION POINT.

Can we construct the station points? Given our eye is at F, and given an epicycle with center E, radius EC, and given that

EC : CF > epicycle speed : star speed

can we construct the line FGB so that

½ BG : FG = epicycle speed : star speed ?

We could employ some sort of Dedekindian postulate, and say to ourselves "the ratio of the half of the intercepted segment to the external segment decreases continuously as FGB goes away from E, so there must be a place in between where the proportion holds good."

But it turns out we can actually construct it.

Given: e : s is a given ratio (expressed in numbers or lines)

To do: Find a secant FGB so that
½ BG : GF = e : s

First cut EF at H so that

EH : HF = e : s [Euc.6.10]

Next, describe a circle on HF as diameter, cutting circle E at G.
Describe a circle on EF as diameter.
Now join FG, cutting the circle on EF at P, and cutting circle E at B.
Join PE.
Join GH.

Now,	∠EPF is 90°	[angle in a semicircle]
and	∠HGF is 90°	[angle in a semicircle]
so	PE is parallel to GH	
so	PG : GF = EH : HF	
so	PG : GF = e : s	[we cut EH : HF = e : s]

But EP is perpendicular to BPG, and E is the center, hence BP = PG,

hence PG = ½ BG

so ½ BG : GF = e : s

Q.E.F.

PORISM: Since the construction is possible only if the circle on HF *cuts* the circle E, H must be *inside* circle E, and therefore EC : CF > EH : HF, i.e. EC : CF > e : s, and hence this is a condition for the possibility of the construction.

WAIT A SECOND—WHERE'S THE EQUANT?

Ptolemy appears to have ignored the equant, and even the eccentricity of the deferent, for these initial demonstrations. That is, he assumes that the center of the epicycle makes its mean movement about F, our eye, when in fact it does not. This is because he is so far only considering the matter abstractly and in the simplest case. It is only in Ch. 2 that he begins to study the particular planets, and to apply the general theorem to the model for Saturn (for example), to find out where it makes stations, how long it should spend in regression (depending on the place in the zodiac etc.).

First he takes it at its mean distance, where the mean movements in longitude and anomaly will be very nearly the same as the apparent movements. So that is like the ideal case already done, but with the particular numbers for Saturn's speeds in L and A.

He finds the angle about the center of the epicycle subtending the semi-arc of the regression, and from knowing the mean speed in anomaly (by the tables generated from the periodic joint returns), he knows the time it takes Saturn to move through double that angle, and so figures the regression, when Saturn's epicycle is at its mean distance from Earth, should take about 138 days.

HOW ON EARTH DID THEY FIGURE THIS OUT?

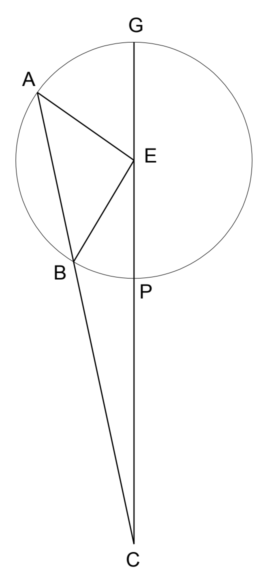
How did Ptolemy's predecessors discover this rule for where station occurs?

One way could be by observations. We can observe the star at station (B) at a given time. Knowing when it is at station, by our tables of planetary motion, we can also know how far the star has gone from its last time at apogee G, i.e. we know ∠GEB. By planetary theory, we know the magnitude of the epicycle, i.e. we know BE : EC.

Since we know ∠GEB, we know its supplement, ∠BEC. Hence we know ∠BEC and the ratio BE : EC, and therefore △BEC is solved, and all its sides and angles are known.

Therefore we also know ∠EBC, and so too its supplement, ∠EBA. But EB = EA, and hence ∠EBA = ∠EAB, and so we know ∠EAB, i.e. ∠EAC.

Thus we know ∠ECA [from △BEC]
and we know ∠AEC [by Euc.1.32]
and we know the ratio AE : EC [the magnitude of the epicycle]
and therefore △AEC is solved, and all its sides and angles are known.

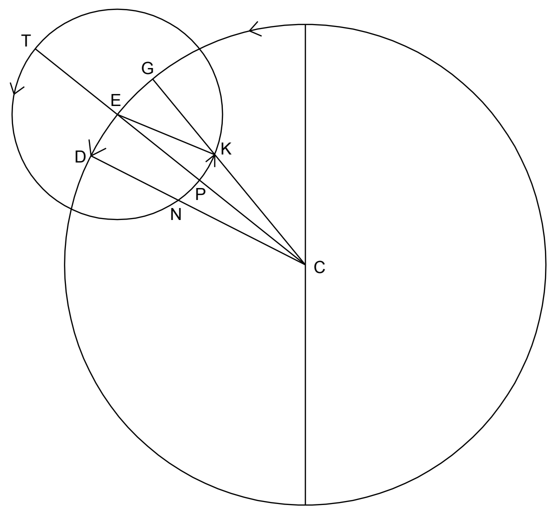


Therefore we know AC : EC [from △AEC]
and we know BC : EC [from △BEC]
so that, putting EC in the same terms for each ratio,
we know AC : BC
so we know AC – BC : BC as well,
i.e. AB : BC
is given.

Comparing this value (for each planet) to the ratio of the epicycle's speed to the star's speed, we notice in each case that

½ AB : BC = epicycle speed : star speed

IDEM ALITER. But since it is very difficult (or impossible) to observe a station accurately, i.e. to say exactly when a star is at station (since it is really just an instant in time, and just seems longer because the planet is moving so slowly for a while on either side of station), this is not likely to be the way the ancients discovered the rule. Perhaps they discovered it by beginning with a simple case, e.g. when there is only 1 station, namely at P, perigee, and by thinking (if this is not anachronistic to say) Newton-style. We know that if a station occurs at P, then the speed of the star and of the epicycle must be such that, as we take equal arcs NP, PK smaller and smaller,



$$ep.speed : star\ speed\ \ u = \frac{arcED}{CE} : \frac{arcPK}{EP}$$

$$sp.speed : star\ speed\ \ u = \frac{ED}{CE} : \frac{PK}{EP}$$ (chords ult. as arcs)

$$sp.speed : star\ speed\ \ u = \frac{NP}{NC} : \frac{NP}{EP}$$ (ED : CE = NP : NC, and NP = PK)

$$sp.speed : star\ speed\ \ u = EP : NC$$ (ratio of inverses)

so ep.speed : star speed = EP : PC

since NC is ultimately equal to PC, and since (the ratio EP : PC being fixed, and that of the speeds also being a fixed ratio) there is no longer any reason to say "ultimately."

But EP is half of PT. So this might lead to suspicion that this is an instance of a general rule, which could easily be verified by a procedure like the one above, looking to a pair of symmetrical stations, instead of just one of them at P.

QUESTION: Does the star ACTUALLY sit still in space (i.e. have an instantaneous velocity of zero)? Or does it just appear to be still, when in fact it is heading right for us or straight away from us? In the simple case of one station, where it happens at perigee, it seems to be a real standstill. But maybe not in the more general case.

COPERNICUS

DAY 27

TYCHO BRAHE'S SEMI-HELIOCENTRIC MODEL

We are now transitioning to the thought of Copernicus. Oddly, however, a very good way to transition to the heliocentric vision of Copernicus is by way of the semi-heliocentric model of Tycho Brahe, who came after Copernicus. Although this is a bit out of historical order, Brahe's model really is a halfway house from full-blown Ptolemaic geocentrism to full-blown Copernican heliocentrism. So here in Day 27, we will spend a little bit of time on this transitional figure.

THE FASCINATING AND STRANGE TYCHO BRAHE

Tycho Brahe was a Danish astronomer who lived from 1546 to 1601. Though that may make him sound boring, he was really anything but. He was born the only son to a pair of high-ranking Danish aristocrats who for some reason (possibly drunkenness) promised to hand him over to the husband's childless brother. When Tycho was born, however, his parents showed no intention of handing him over, and the brother said nothing, so they just kept him—until he turned two, that is, and the brother swooped in and claimed him. Strangely, the parents did not put up a fight, but handed the boy over. It doesn't do for aristocrats to go back on their word, apparently. And so Tycho was moved from one castle to another.

Tycho's uncle (and foster father) Jørgen Brahe intended for Tycho to study law and philosophy, but a solar eclipse in 1560 inspired him to become an astronomer instead. While he was at the universities of Copenhagen and Leipzig, whenever his tutor dozed off, he would put down his law books and observe the stars. In those early days he had no more equipment than a globe and a pair of compasses, but that was sufficient to enable him to detect significant errors in the Alfonsine and Prutenic tables then used for developing the calendar, and to correct them.

When he was 20 years old, Brahe got into a mathematical disagreement with another Danish aristocrat by the name of Manderup Parsbjerg. Naturally, this led to a duel—in which Brahe lost a good chunk of his nose. For the remainder of his life he wore a prosthetic one made of metal. It was for a long time thought to be gold, and possibly he did have a golden fake nose, but apparently when his body was exhumed in the 20th century the nose with which he had been buried turned out to be copper. That is odd, since he was certainly wealthy enough to have a gold nose (as we shall see). Some have speculated that a copper nose would be much lighter, and therefore more comfortable than a golden one. Or did someone steal his gold nose just prior to his burial? No one knows.

Tycho Brahe discovered a "new star," as we read in Arthur Koestler's *Sleepwalkers*: "On the evening of 11 November, 1572, Tycho was walking from Steen's alchemist laboratory back to supper when, glancing at the sky, he saw a star brighter than Venus at her brightest, in a place where no star had been before. The place was a little to the north-west of the familiar 'W'—the constellation of Cassiopeia, which then stood near the Zenith. The sight was so incredible that he literally did not believe his eyes; he called at first some servants, and then several peasants to confirm the fact there really was a star where no star had any business to be. It was there all right, and so bright that later on people with sharp eyes could see it even in the middle of the day. And it remained in the same spot for 18 months." It was, in fact, a supernova. Unlike a comet, it stayed in the same spot every night among the stars.

Publishing and lecturing on this new star won Tycho immediate fame as an astronomer, and he was in high demand everywhere. The King of Denmark and Norway, Frederick II, wanted to keep this prize at home, however, and managed to do so basically by giving Brahe an enormous bribe. In return for serving as the court astrologer, Brahe would have the island of Hven for his own on which to build an observatory and a castle, and he would also have money enough to equip these as he saw fit. The total amount of money going his way is estimated to have been about 5% of Denmark's GNP at the time– Not bad. Brahe accepted this arrangement, and in 1576 built the castle of Uraniborg ("fortress of the heavens"). The place had a large number of furnaces for conducting other kinds of experiments, too, chemical and medical. It was something of a prototype for European research centers. As the telescope had not yet been invented, Tycho was still conducting naked-eye astronomy, but taken to the limit. He spared no expense in having made gigantic and very finely made instruments to assist in his observations of the heavens. For twenty years Brahe made his observations there.

When King Frederick II died, however, his 19-year-old son, Christian, took the throne, and for some reason Brahe fell entirely out of favor. Some people believe that Brahe had had an affair with Frederick's wife, Christian's mother, and that this was known to Christian. Some people even suspect that Brahe was murdered. He died at a banquet, supposedly of a bladder problem, but when his body was exhumed traces of mercury were found on the body. Was the mercury something he had been medicating himself with for a long time? Or something he inhaled during his chemical experiments? Or was it put into his cup at his last supper? No one knows for certain. Some have speculated that Brahe was part of the inspiration for Shakespeare's *Hamlet*. A Danish adulterer with the queen, a young and vengeful heir to the throne, and mercury poisoning.

Whatever the cause, when his first patron died in 1588, Tycho fell out of favor, and all his former privileges were revoked. Christian IV even withdrew the precious observatory from Brahe, although it could hardly have been put to better use in anyone else's hands (except perhaps Kepler's). But the Holy Roman Emperor invited Brahe in 1597 to build a new Uraniborg (and to live on a pension of 3000 ducats) upon an estate near Prague, in Benatky nad Jizerou. Brahe died before it was completed.

Brahe himself was married to a commoner, by whom he had eight children. He kept an elk as a pet. Once, while it was away paying some sort of visit to another aristocrat, it got drunk on beer, fell down a set of stairs, and died. Brahe also had a "little person" as a court jester whose name was Jepp, whom Brahe believed to be clairvoyant.

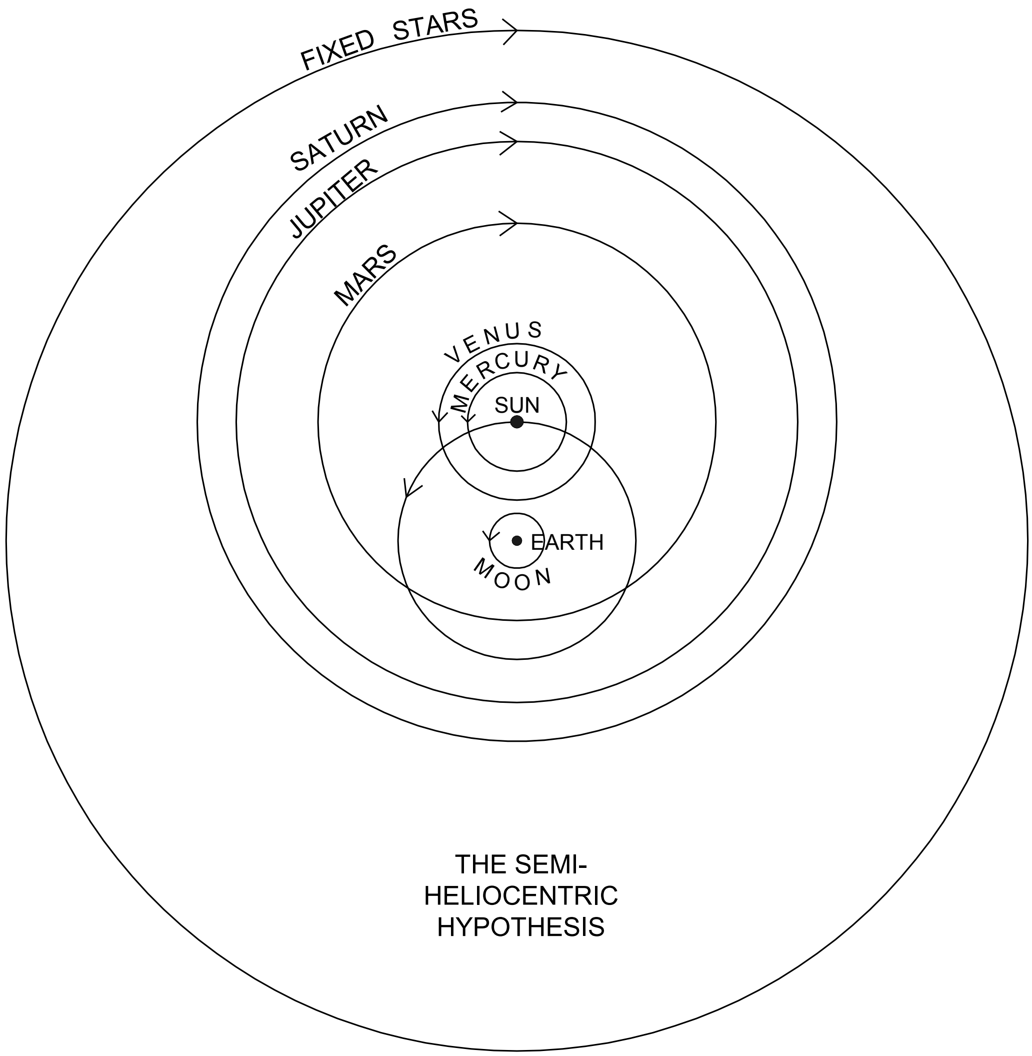
Brahe's moustache was apparently quite long and eccentric. The odd details of his life seem to be endless. Interesting as these are, we are more interested in his work.

Some of Brahe's writings are *De Nova Stella* ("Concerning the New Star", 1573), *Epistolae Astronomicae* ("Astronomical Epistles", 1596), and *Astronomiae Instauratae Mechanica* (a work in which Brahe describes his equipment, 1598).

Johannes Kepler served as Brahe's assistant in the years 1600 and 1601. When Brahe died, Kepler took full advantage of the confusion and more or less stole Brahe's records of astronomical data. In life, Brahe had been extremely possessive of this data, and did not even permit Kepler to look through it freely. (Some people have speculated that Kepler murdered Brahe for the data—but there is no real evidence of that.) Kepler thus came into possession of many tables of extremely precise data with which to compute the true orbit of Mars, and with which he would discover that the planets move on elliptical paths. Brahe's obsession with precise observations, and the later fruit of this arduous labor in the discoveries of Kepler, ushered in the age of precision in astronomical measurement. Brahe took this as far as naked eye astronomy could go. He determined the length of the solar period to within less than a second, which led to the abandonment of the Julian calendar in favor of the Gregorian.

Kepler was familiar with Brahe's precision and accuracy, and took Tycho's observations as infallible and unquestionable. If Brahe's observations disagreed with the Ptolemaic hypothesis by one arc-second, that meant the Ptolemaic hypothesis was simply wrong.

For now, however, we are mainly interested in Brahe's own model of the planetary motions. Tycho Brahe never accepted the Copernican model, in which the planets, including Earth, all go round the sun. Brahe instead supposed that the planets all revolve around the mean sun, while the mean sun orbits an immobile Earth. This theory was made possible by supposing that the mean sun was inside the epicycles of the inner planets (Venus and Mercury) and also inside the moving eccentrics of the outer planets (Mars, Jupiter, and Saturn).



Note the following in this diagram of Brahe's Semi-Heliocentric Hypothesis:

• The moon goes around the Earth.

• The planets all move on epicycles which go around the mean sun at their common center.

• The inner planets are on same-direction epicycles which are smaller than the deferent, i.e. than the sun's orbit around the Earth.

• The outer planets are on opposite-direction epicycles which are larger than the deferent, i.e. than the sun's orbit around the Earth—hence they can also be thought of as moving eccentrics spinning about the Earth. The Earth is inside the epicycles of Mars, Jupiter, and Saturn, and so these are really moving eccentrics for us, and so the stars must move on these in the opposite direction of the eccentrics themselves.

• Earth remains *immobile* and at the center of the universe.

• The planetary orbits are drawn roughly to scale (but the sphere of fixed stars is drastically diminished to fit it onto the page).

How do the speeds S, L, A enter in?

• The epicycles of Venus & Mercury move with longitudinal speed L = S, and each moves on its epicycle with speed A.

• The eccentrics of Mars, Jupiter, Saturn move with speed S = L + A, and each moves "backward" on its eccentric with speed A.

COMPARISON TO PTOLEMY.

What is *new* in the diagram? How does it differ from Ptolemy? Really there is only one feature distinguishing this model from the Ptolemaic system: The sun (i.e. the mean sun) is *inside* all the epicycles, i.e. the sun is at the center of all planetary orbits. Perhaps there is also this: Ptolemy never really presented us with a single diagram of the universe as he understood it. This is partly because he saw that there were equivalent models for each of the motions, and partly because he did not know for sure where to place the sun in the order of the spheres. But surely it is desirable to have a picture of the universe! Brahe is providing us with one here.

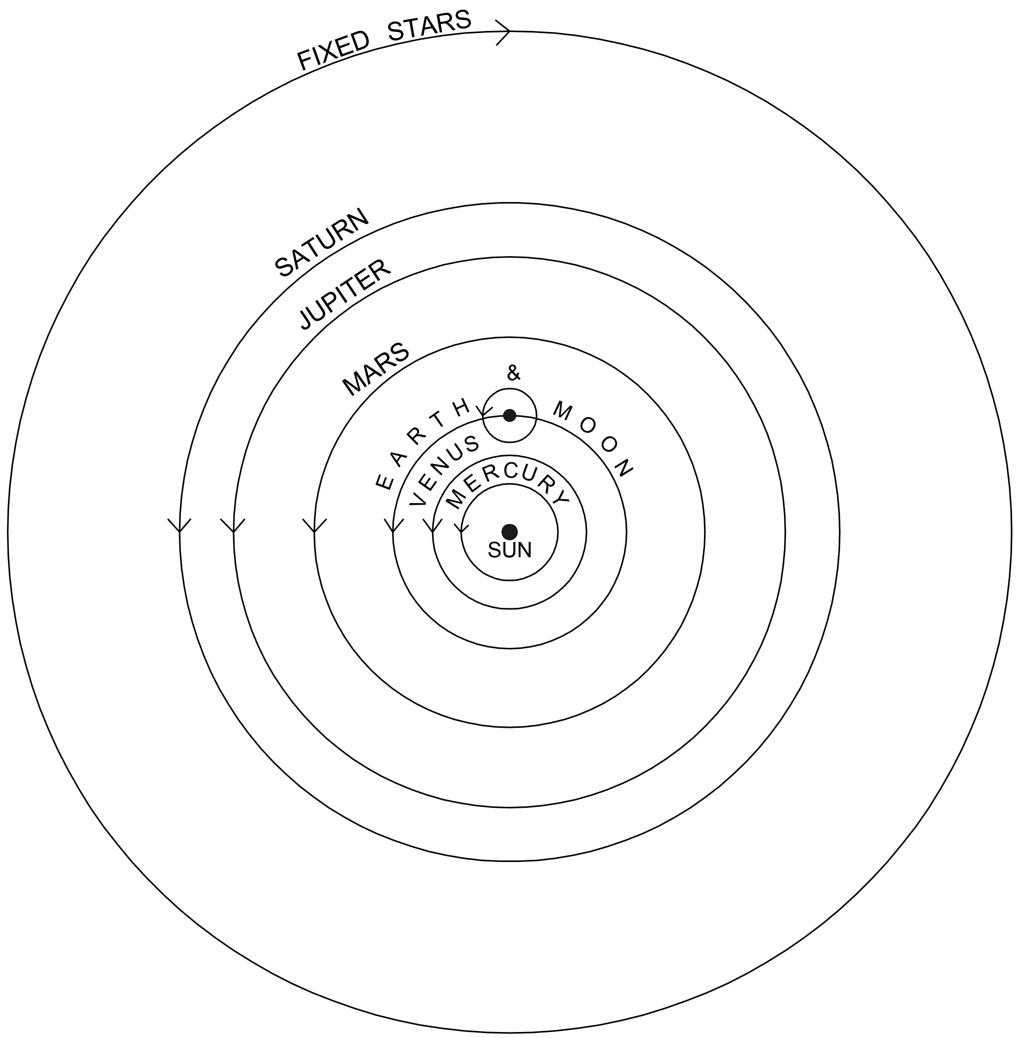
ADVANTAGES.

Unlike the Ptolemaic universe, the Brahean is unified. All the planets (except Earth, which to them was not a planet) are orbiting the sun, rather than making epicycles at unspecified distances from Earth. And since the sun orbits Earth, the Earth-Sun distance becomes a common element in every planetary model, enabling us to compare all the distances to one another.

DISADVANTAGES.

- Brahe still has everything orbiting the Mean sun, a mathematical fiction.
- It is not easy to conceive of a physics that would account for these motions: Why are so many things going around the sun? Is it because it is so big and influential? Then why isn't the Earth going around it?

• If we make one slight change, and say the Earth is orbiting the sun, and it is the sun that is sitting still, then we get the diagram below, i.e. the full-blown heliocentric and Copernican model, where everything orbits the sun in concentric circles and in the *same direction*! And the epicycles of Venus & Mercury become mere orbits. As to the appearances, these two models can be made perfectly equivalent. The diagrams themselves seem to have already made a powerful argument in favor of Copernicus already.



THE
HELIOCENTRIC
HYPOTHESIS

COPERNICUS

DAY 28

INTRODUCTION TO COPERNICUS (1473-1543)

BRIEF BIOGRAPHY.
Nicolaus Copernicus was born February 19, 1473 in Torun, Poland. When his father died, his maternal uncle, Lucas Watzelrode, adopted him and his three siblings. This uncle was a scholarly priest, and was later made the Bishop of Ermland in 1489. He intended for Copernicus to be trained for the church. Copernicus began study at the University of Cracow in 1491 where he met and studied under Albert Brudzewski, an expert in Ptolemaic astronomy and leader of the humanist movement at the university. Under him Copernicus developed a love of mathematics and astronomy and left Cracow without yet having obtained a degree. But his uncle then provided him with a living without significant conditions attached, leaving Copernicus more or less free to travel and study. He went to the University of Bologna, and continued his study of canon law in preparation to work for the church as a canon. He was appointed canon of the cathedral of Frauenburg in 1497, but immediately took a leave of absence to continue studying. From 1501 to 1505 he studied medicine at the medical school at Padua (except for a short period in 1503 when he finished his doctorate in canon law). In 1506, when he returned to Poland, he was a humanist learned in Greek, mathematics, astronomy, and he was also a jurist and a physician. He became physician to his uncle, the bishop, and lived with him in the palace of Heilsberg. He was involved in the reconstruction of Ermland after the war between Poland and the Teutonic Knights had ended in 1521. He was uninvolved, however, in the Reformation.

Copernicus first began to develop his own astronomical theory in 1506, but it was a long time before it came into publication. He pursued astronomical observations at an observatory he himself had established in Frauenberg. In 1514 he was invited by the Lateran Council to give advice on the reform of the calendar, but he declined. Only in 1530 did Copernicus first begin to publish his heliocentric ideas—first in the *Commentariolus*. His ideas attracted a great deal of attention, as one would expect, but apparently did not stir up the sort of controversy one would expect, given the fiasco involving Galileo in the next century. In Rome, people lectured on Copernicus' new ideas, and Pope Clement VII gave some measure of approval. It was not until after 1540, however, that Copernicus was convinced by his friends to allow Joachim Rheticus to publish the *De Revolutionibus Orbium Coelestium* (the work we will be studying in this course). Copernicus was shown an advance copy of his book on May 24, 1543—the very day he died.

THE TITLE OF COPERNICUS' WORK.

The revolutionary book we will be drawing from is *De Revolutionibus Orbium Coelestium*—"On the Revolutions of the Heavenly Spheres." Copernicus was still a believer in the crystalline rotating spheres in the heavens, although he had a very different idea from Ptolemy about how they were ordered and how they moved.

PREFACE "TO THE READER".

At the outset of the book, there is a little address "To the Reader" which was certainly not written by Copernicus, but perhaps by a certain Andrew Osiander, a Lutheran theologian and friend of Copernicus. It is certainly not written in the same spirit as *De Revolutionibus* itself! Its main claim is that an astronomer can neither prove the truth of any of his hypotheses, nor find out the true causes of the celestial appearances, nor can he get any probability about these things. It is his sole job to "save the appearances" by the simplest model possible. He is an artist *as opposed to* a scientist.

What motivates this preface "To the Reader"? Putting the Earth in motion and the sun at the center of the universe ran contrary to the established scientific view, but to many people at the time it also seemed to run contrary to the Bible, because in various places the Bible seems to put the Earth at the center and speak as if the sun moved around the Earth. To defend Copernicus from too nasty an attack, then, or perhaps to prevent attacks on the Scriptures, or avoid conflicts between astronomers and theologians, Osiander opens with a disclaimer, as though to say: We do not say this is true, or even probable—just a *simple* way to save the appearances, nothing more.

Osiander wishes to give evidence for this agnostic view of astronomy, that it can attain to nothing certain or even probable. One example he employs is supposed to be paradigmatic or sufficiently representative: the Epicycle of Venus. It is so big, compared to its eccentricity, that Venus would appear about 16 times bigger near perigee than when it is near apogee, which is against the facts (we saw this back in Day 23).

This is just as much a problem for Copernicus as for Ptolemy, although it is solved by the phases of Venus, if one makes Venus's epicycle circle the physical sun as its center (or nearly its center).

But what is the purpose of astronomy according to the author of this preface, if not to find the truth and the real causes of celestial motions? Answer: To "provide a correct basis for calculation," that is, in order to be able to make accurate predictions about where and when things will appear in the heavens. But to what end? Not astrological (since the author is a Christian). Then for what—navigation? To make calendars for the Church? Why should we despair of ever attaining to any real understanding of what is going on in the skies above us? Osiander (or whoever authored this preface to *De Revolutionibus*) seems to be motivated too much by fear of conflict between science and faith.

PREFACE & DEDICATION TO POPE PAUL III.

Copernicus actually did write the preface and dedication to Pope Paul III. Does it say the same thing as the Preface "To the Reader"? Does it endorse a complete agnosticism about the truth concerning celestial appearances?

No. Certainly he is cautious, but he seems more anxious to defend the reasonability of his hypothesis, even its superiority to the old view, which seems to make the universe into a strangely unintelligible heap of disconnected things. He finds fault with the old view not so much for being inaccurate (in predicting where things will be when), but for being incoherent, impossible, full of coincidences etc. This sounds like a concern for truth!

What, specifically, led Copernicus to begin astronomy afresh? He tells us himself:

a. Disagreement among experts.

b. Disagreement with the phenomena.

c. Inconsistency of principles.

d. Inability to explain the most desirable quantities (and a decided lack of unity and harmony and simplicity in the whole system).

More specifically:

a. "Mathematicians have not agreed with each other," he says. This is the first sign of a problem. "They do not assume the same principles, assumptions, or demonstrations for the revolutions and apparent movements." (If the "experts" cannot agree, then why shouldn't Copernicus be allowed to propose his own ideas?)

b. "They have not been able to establish anything for certain that would fully correspond to the phenomena," he adds. None of the existing theories exactly predict the motions of the heavenly bodies.

c. "They have in the meanwhile admitted a great deal which seems to contradict the first principles of regularity of movement." We saw an instance of this even as far back as Ptolemy: the equant. The center of the equant-circle was not the center of the actual circular track that the epicycle's center moved around, but was only the center of the uniformity of angular velocity. Copernicus wants to be as consistent as possible, and make heavenly bodies move only on circles whose centers are the points around which they move uniformly. So we are still hanging on to that old "Astronomer's Axiom," and if anything, we are clinging to it a bit more than we did with Ptolemy!

d. "They have not been able to discover or to infer the chief point of all, i.e. the form of the world and the certain commensurability of its parts." Ptolemy did not give us a single picture of the universe, nor did he tell us what the ratios were between the distances in the various planetary hypotheses. How big is Mars's eccentric compared to Jupiter's? Ptolemy will not commit himself. So his theory is really like a dismembered body rather than a unified whole. But surely the universe is a unified whole. So something crucial is missing from the picture.

Copernicus suggests that his system, as soon as he assumed that the Earth's rotation accounts for the daily movement, and its orbit around the sun for the sun's apparent yearly motion, immediately committed him to many things and decided many things for him. He could not play around with it and do as he liked the way Ptolemy could. He proposes this, it seems, as a sign of the truth of his system. It exhibits a certain necessity with which everything falls into place.

WHAT'S TYCHO'S PROBLEM, ANYWAY?

Looking back to Tycho Brahe's semi-copernican hypothesis, one might wonder why he (or anyone else) would hesitate to adopt the Copernican model. The two models, i.e. the semi-heliocentric and the full-on-heliocentric, can be made entirely equivalent as to the appearances. So why not prefer the simpler Copernican view?

For one thing, there were physical problems, in particular not feeling the movement of the Earth, whereas Copernicus requires the earth to be in motion—both spinning on its axis to explain the daily motion in the heavens, and again orbiting the sun once a year to explain the sun's apparent annual motion.

For another thing (and possibly this was the bigger issue), to make the earth move around the immobile sun at the center is to make the earth a planet, i.e. a star, i.e. a celestial body, which is to make the celestial bodies terrestrial. The whole idea of "the heavens" changes, the whole world-view changes. The heavens are brought down to Earth, and Earth is placed in the heavens. The heavens are no longer other-worldly, made of incorruptible quintessential stuff. Instead, they are Earthy, like Earth. A great deal of cosmology had been built on those old ideas, and no one seemed ready to replace them with anything.

For still another thing, Copernicus still has things moving on solid crystalline spheres. But Tycho Brahe correctly showed that this was impossible by showing that the trajectories of comets took them right through the spheres of the various planets! Why didn't the comets smash holes in the spheres? Or why didn't they themselves smash to bits against a sphere?

DE REVOLUTIONIBUS, 1.1-2, 1.4-5

Now we come to the book itself. Copernicus opens with an introduction to the work in the classical style, in which he outlines the dignity, the place, the utility, and the difficulty of his science or art.

He calls it the "head of the liberal arts," on the grounds that its matter is superior to that of the other liberal arts, and on the grounds that the other arts are in its service (or used by it). Certainly the ones dealing with words (grammar, rhetoric, logic) deal with a matter which is more human, and less divine, and less worthwhile in itself. And if arithmetic is studied to some extent for the liberal art of music (which is to some extent studied as preparation for ethics and political philosophy), and geometry is to some extent for astronomy (which is to some extent studied as preparation for natural philosophy and metaphysics), in the order of the liberal arts toward philosophy and theology, then Copernicus seems to be largely right.

Chapter 1

In Chapter 1 of Book 1 of *De Revolutionibus*, Copernicus assumes that the universe is spherical.

Is this plausible? His reasons are largely from fittingness, or from precedence. For example, this is the shape of the sun, the moon, the earth, and he even says we find a precedent for this shape "in the case of drops of water and other liquid bodies, when they become delimited of themselves." That is interesting—he is already thinking more physically than Ptolemy did. And very inductively (as Newton will do later).

The earth is a (near) sphere because of the downward tendency of all its parts to the center. This shape is the effect of gravity. But a soap bubble in the air is a (near) sphere because of the random bangings of air molecules inside the film of soap—there is no more banging in one direction than another, so one gets a sphere, the shape of indifference.

This chapter certainly does not settle the question. But if the universe is finite and bounded, then the sphere seems to be the simplest shape, and to be a shape which results naturally (or at any rate automatically) from many different principles out there. So it is plausible.

Chapter 2

In Chapter 2, Copernicus is arguing that the earth is a sphere.

a. One piece of evidence: as you travel north, the North Celestial Pole moves more and more overhead.

b. More specifically, in places equidistant from the North Terrestrial Pole (same latitude) the Celestial Pole will cut the meridian in the same ratio (or, more simply, will have the same height above the horizon).

a. & b. are supposed to prove Earth is a sphere going from north to south.

c. If you see an eclipse (solar or lunar) in the evening, i.e. near the west, people in the east do not see them; which shows Earth bulges up or is convex in between you and them. So Earth is round going from west to east, too.

d. And you can see land from the top of a ship's mast before you can see it from the deck; and again land appears to rise up out of the sea or sink down into it, which all shows that the water bulges up in between, so that the ocean is convex. But water wants to be flat, or does not have any more convexity than the surface it rests on. So Earth is round.

Note that these arguments are basically the same as the ones which Ptolemy advanced for the spherical shape of Earth.

In Chapter 3, Ptolemy pursues the sphericity of the Earth a bit more. He mentions America in that chapter, which he says geometrical reasons compel us to believe is "diametrically opposite to the India of the Ganges." He mentions the erroneous opinions of various ancients concerning the shape of the Earth—Empedocles thought it was a plane, Leucippus that it was a "tympanoid," Aniximander thought it was a cylinder, and so on.

Chapter 4

In Chapter 4, Copernicus is arguing that celestial bodies move uniformly and everlastingly in circles. This is the old "Astronomer's Axiom" which Ptolemy and Aristotle and Plato also accepted, long before Copernicus.

What support does he give for this idea?

That they move in circles he assumes follows from their shape, i.e. that they are spherical. But that's a bit weird: the earth rotates, and that is in keeping with its shape, but then the earth is not a natural body, but a bunch of natural bodies crammed together, it seems, so it does not really have a "natural motion" of its own, does it? Even weirder, the earth must now rotate around the sun—but what has that motion to do with the shape of the earth? *He is still a believer in heavenly spheres* (hence the title of the book), in which the stars are fixed. So it seems we are to believe that the earth is embedded in a crystal sphere. Already there are theoretical weaknesses here.

He describes some of the basic motions; the "daily" motion from east to west, the "antagonistic" motion from west to east—that of the moon giving us the month, that of the sun giving us the year (about the poles of the ecliptic).

He insists on the Ptolemaic principle of perfect circles and perfect uniformity of speed, "since it is impossible that a simple heavenly body should be moved irregularly by a single sphere", "For that would have to take place either on account of the inconstancy of the motor virtue" (God or an angel or the soul of a heavenly sphere getting tired or sloppy, or else the celestial Nature not always being the same) "or on account of the inequality between it and the moved body" (some inequality or inconstancy in the degree to which the same power is effective on the body, as though it got closer to the power sometimes, and further at other times—which is absurd if the power is incorporeal, or is the nature of the body itself). But none of that will do—"since the mind shudders at either of these suppositions." That is supposed to be the "best system" up there, after all!

Hence, if there appears to be any irregularity of speed or direction, this is merely apparent, and can be explained as a property of the place from which we are observing the motion.

So he does not differ substantially from Ptolemy in this regard.

But in a way he is already not entirely consistent with himself—he wants us to believe in the perfect uniformity of motion in bodies which are no longer divine and immortal, or not any more than the earth is!

If Plato, Aristotle, and Ptolemy could ever have become convinced that Earth is moving around the sun and is itself just another one of the planets, so that those "divine wandering stars" are no more "divine" than Earth is, then they probably would have started seriously re-thinking the notion that things up there move in perfect circles and at perfectly uniform speeds. And if they also knew what Tycho Brahe knew, that the comets move *right through all the supposed invisible spheres carrying the planets around*, they certainly would have started all their thinking about the heavens over again.

Chapter 5

In Chapter 5, Copernicus gives reasons for saying (or at least considering the possibility) that the Earth has some motion:

a. In principle, an apparent motion of X can be due either to the motion of X, or to the motion of the observer of X, or to both. (Imagine being on a train next to another train—sometimes it is hard to tell whether you are moving or the train next to you is moving or both.) Now by the daily motion the whole universe seems to be carried around. So if we say instead this motion really belongs to the earth, in the other direction, we would see just what we see up there, with regard to the daily motion of the sun, moon, stars, planets. (Ptolemy himself admits that this is simpler; instead of giving, coincidentally, as it were, the same daily motion to all those things, we just give it to the earth.)

b. The universe *contains* (it is a place), the earth is *contained* (it is in place). Why not give the motion to the thing in place, to the contained, rather than to the container? (One could take this a step further: for the earth to spin makes sense, since it has a reference outside it. But what would it mean for the universe to "spin" if there is nothing outside it which is the immobile reference for that spinning? This might not be unintelligible, but it is odd.)

c. Some ancients attributed daily motion to the earth, and some even said that the earth is a planet.

COPERNICUS

DAY 29

BASICS OF COPERNICAN THEORY

De Revolutionibus, 1.6-8

Chapter 6

The title of this chapter is: *On the Immensity of the Heavens in Relation to the Magnitude of the Earth*. This is reminiscent of Ptolemy's *the earth is as a point to the heavens*. But Copernicus is bringing this up to show how this immensity opens the door to the idea that the Earth might in fact have a motion other than just spinning, e.g. it might have an *orbit* around the sun, and the sun might be at rest in the center instead. And that orbit might be small enough, compared to the sphere of fixed stars, that it would not enable us to detect any (naked-eye) parallax on them.

Copernicus repeats the main argument for the immensity of the heavens compared to the Earth:

Since at one time we see six signs but not the other six, then at another time we see the other six but not the first six, therefore each group of six forms a semicircle of the ecliptic, the great circle on the Celestial Sphere (in which he still believes, although it no longer has a real job to do! The circular motion of the stars was the main reason to suppose that the stars were on a big sphere, and that motion is about to be taken away from them for good). Therefore, when we see the first six, we are on a diameter of the ecliptic, and when we see any other six we are on a different diameter of the ecliptic, and therefore we are at the *center* of the ecliptic, and therefore at the center of a great celestial circle. Therefore our horizon is a great circle.

But this is equally true all over the earth, and so there is no (sensible) difference between the horizon through my eye and the plane parallel to it through the center of the earth. Therefore the earth has no appreciable size compared to the celestial sphere.

Copernicus says next that *it does not follow that the earth must be at rest at the centre of the world*.

One might think it would follow, since along the way we said we are at the center of the universe, i.e. on two diameters of the celestial sphere. How can we wander from the center?

But precisely because the earth is so tiny, just as all these phenomena "are equally true all over the earth," so too, perhaps, because its orbit around the sun is so tiny (compared to the celestial sphere), all these phenomena are equally true at any point along Earth's orbit. Copernicus is not quite this explicit, but it seems to be what he is driving at.

In this chapter, Copernicus also seems to be addressing the view that the Earth cannot be moving very much *precisely because it is close to the center of the universe*, as though it were a grain of sand near the center of a vinyl record, so it moves very little in the same time as things near the rim move a lot. But in that view, the "daily motion" would still be a movement of the whole universe, not Earth's rotation—so we must imagine Earth *not* spinning, but making a tiny orbit each day, and he says it is as clear as "daylight [pun intended!] how false that is," since there would always be noon in one place on Earth, and always midnight at another, and the sun would not rise and set (so that the sun and the earth would be like two spots on a record).

And he goes on: since the planets and their spheres all have natures different from each other, therefore the speeds are not all of one angular velocity, but the inner planets, for instance, go around the zodiac once a lot faster than the outer ones do.

Chapter 7

The next chapter is about ARGUMENTS FOR A STATIONARY EARTH.

Here Copernicus recounts some Ptolemaic and Aristotelian arguments for geocentrism (which stands or falls with Earth sitting still or moving).

(a) Ptolemy's first argument for a stationary Earth was that what is heavy endeavors to get to the center and rest there.

(b) The next argument for an immobile Earth is based on the Aristotelian view that *the movement of a body which is one and simple is simple*, and the simple movements are rectilinear and circular. And of the rectilinear type of movement, one is up, the other is down. But earth and water ("simple" bodies), which are heavy, tend down, i.e. to the center. So it seems fitting to give the four elements rectilinear motions, and the heavenly bodies circular ones.

(c) The third argument is based on the Ptolemaic view that "things which are suddenly and violently whirled around are seen to be utterly unfitted for reuniting, and the more unified are seen to become dispersed, unless some constant force constrains them to stick together."

Basically, since you feel it when you are whipped around in a circle, as on a merry-go-round, you would feel it all the more if Earth whipped around at a 1000 mph. Things would fly off and apart with such a speed of such a prodigious body. But you don't feel that. So Earth is at rest.

Chapter 8

REFUTATION OF ARGUMENTS FOR A STATIONARY EARTH

Copernicus now addresses arguments such as these.

PTOLEMY'S VIOLENT ARGUMENT. Against (c) above, Copernicus says that things would *not* be thrown apart by the circular motion of Earth if that motion is not violent, but natural, and hence a movement which all contribute to and partake in smoothly. Or, if Ptolemy denies this, why did he feel no anxiety about the universe itself being torn apart by its even greater speed?

(There is something to this. Something must be capable of moving with a great rotation without flying apart, whether it is the earth or the heaven. But there are weaknesses, here, too. The earth is made of stuff which we know *can* fly apart. When you spin mud, it likes to fly off on straight lines along the tangents, it does not "like" to move in a circle. So this reply of Copernicus does not adequately explain why we do not feel the motion of the earth. Also, if "earthy stuff" has a "natural motion," isn't it the motion *in a straight line down*? So if we also say the earth has a natural motion in a circle about its own axis, aren't we saying it has two natural motions at once? In fact, don't we have to add at least one more, since the earth also, on this view, will be orbiting the sun? Well, then, does the *simple* body of the earth have *two* natural motions? We will see how Copernicus will address this below.)

He mentions that someone might reply to him that the heavens *do* fly apart, and that is why they are so spread out and huge (as Anaxagoras might have said). He replies: but then the heavens would have to be infinite, since the speed increases the size, and the size increases the speed (since it keeps going round once in 24 hours, regardless of the radius). But what is infinite cannot be traversed, and from this it follows the universe cannot spin, but must be at rest.

(If you think this argument is strange, you are not alone!)

He says we can't really know whether the universe has limits, anyway (interesting), but we know the earth does, and it is round, so why not give it the natural motion that accords with its shape, i.e. moving in a circle, "rather than put the whole" universe "in commotion"?

(Again, there is some weakness in this, since the earth is not a natural body, but more like a collection of them; also, is it true that everything spherical likes to move in a circle?)

He gives an *example*: Sailors feel as though their ship is at rest, and it seems like the land comes to them. Why can't the earth be like that? Its movement is unfelt, and it only looks like other things are moving, but the motion really belongs to the earth?

(That is better!)

PTOLEMY'S CLOUD ARGUMENT. Ptolemy argued that if the earth spun then the clouds would all move one way (in the direction opposite the spinning of the earth). To this Copernicus replies that nothing prevents us from supposing the atmosphere, too, spins with the earth, "and whatever other things have a similar kinship with the earth." He suggests that this is either because the atmosphere, having earthy and watery stuff in it, "obeys the same nature as the earth," or else its motion is "acquired," as if it had been set spinning by the earth.

COMETS. Copernicus is aware that someone might bring up comets (or "bearded stars") as evidence against him. Copernicus is saying that everything "earthy" would share in the spinning motion of Earth, our atmosphere included. But then why, someone might ask, do the comets not spin together with Earth, but rise and set like the stars do? Comets were presumed, even in the time of Copernicus, to be *unheavenly* things, because they were generated and corrupted. Stars and planets never changed, never ceased to be, never came to be, but comets suddenly appeared and then again disappeared. It was presumed they were coming into existence, rather than just coming into view! Hence they were more like Earth than the heavens, and it was presumed that their place was in our upper atmosphere, that is, still in the realm of the generable and corruptible, in the sublunar sphere. Copernicus replies to this by saying that comets are so high in our atmosphere that at that place the atmosphere does *not* share in the spinning of Earth, just as Ptolemy said would happen generally for our whole atmosphere. (The truth is that comets are on very elliptical orbits, and they are relatively small compared, say, to Jupiter, and so they become visible only when they are at the solar end of their orbits, but then they pass out of naked-eye view again for the much longer portion of their orbits. Copernicus seems to think of the heavens as still "heavenly," although that is rather odd if Earth is among the planets moving about the sun! Tycho Brahe saw the implications of this more clearly than Copernicus, apparently, and so he was unwilling to make Earth effectively one of the heavenly bodies. Tycho also discovered a supernova—which proved once and for all that things can come to be and cease to be even among the fixed stars.)

There is another way to verify that the comets are in fact not in our atmosphere. There is a visible parallax of the moon and also of comets—and it is *less* pronounced for comets. Hence they are further away than the moon! And this would have been observable perhaps even by Ptolemy (had he bothered to check), and certainly by Copernicus. At any rate, just after Copernicus, Tycho Brahe observed comets very carefully and showed that their paths *went through the supposed spheres*, and hence destroyed the crystalline spheres idea forever. So it seems that Brahe got around the problem of comets by maintaining that everything up there is celestial, but the comets are not generated and corrupted, they just get closer and come within view. (But how did he account for the supernova, the "new star," which just stayed in the same place? Here he was seeing the death of a star, and yet refused to admit that the stars suffer death!)

Copernicus's RESPONSE TO ARISTOTLE's ideas about natural motion is a bit prolix. He concedes that a simple body has a simple motion so long as it is in its natural unity (integrity) and in its natural place. What he seems to be getting at is this: Yes, when you take a piece of Earth away from its natural whole, the earth, it has a new motion downward, toward the center, trying to get back to the whole to which it belongs—it is "sickly," not as it ought to be, violently shoved away from where it is supposed to go. Hence its motion in that condition is also "sickly": it is not moving as it likes to, but moving back to where it wants to be so that it can move as it likes to. And all rectilinear motion is that way, i.e. an endeavor to get back into the right place or back into the whole. Hence the only truly natural motion, which can belong to a thing when it is in its right place and in its proper whole, is circular motion, by which it does not really move out of its place or out of its whole. So the earth can rotate, since that is not contrary to the "earthy nature," but just what it does in its natural place and in its natural whole (and presumably, too, the celestial sphere in which the earth is embedded has its own natural rotational motion). The earth, then, does not really have two natural motions, but one (i.e. in a circle). The downward motions of its parts are not entirely natural, but the motions they have when they are in an unnatural place.

ADDITIONAL ARGUMENTS. Copernicus now adds this: immobility and rest are more godlike, motion is more earthly and pertains more to mutable things. So, given the choice, we should prefer to assign motion to the earth rather than to the heavens. (Notice again that Copernicus still insists on the distinction between the earth and the heavenly bodies, although he is placing it among them and making it move with them and like them!)

And again: it seems absurd to assign motion to the container (the universe) rather than to the thing contained (the earth).

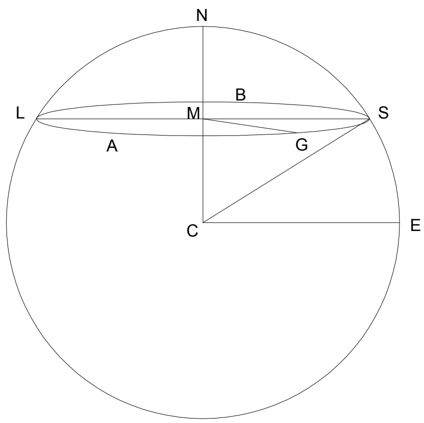
And again: the appearances make it plain that the planets are sometimes closer to the earth and sometimes further away. Hence they do not keep the same distance from the center of the earth (contrary to what Aristotle thought). Therefore they move around other centers besides the center of the earth—which means there seems to be no necessity in saying they move uniformly, somehow, around the center of the earth; but maybe it is enough to say they move uniformly around their own centers.

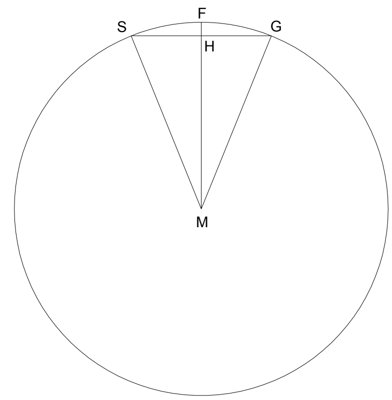
Copernicus finishes with the remark "For all these reasons it is *more probable* that the earth moves than that it is at rest." That does not sound like the agnosticism of the opening foreword "To the Reader"!

WHY DON'T WE FEEL THE MOTION OF THE Earth?

This is a fair question for any of us who agree with Copernicus (as most of us do) that the earth spins on its axis every 24 hours or so. To see the answer well, we would first need to have a big discussion about inertia, and establish that uniform motion in a straight line does not feel any different from rest. Now your motion with the earth, when you stand still on some part of it, consists partly in this inertial component, which feels just like rest, but also it is a little bit curved.

Now, is that curve enough for us to feel?





If you stand on Earth's North Pole, you will turn round once every 24 hours. If I put you on a turn-table that did that, you wouldn't feel it, either—just like you don't see the motion of the hour-hand on a clock. (If it turned once every second you would feel it; your blood would shoot out to your fingertips if you held your hands out.)

But what if you are somewhere closer to Earth's equator? Then you make a giant circle in 24 hours—but take what you do in one minute, and it is mostly straight, and uniform. Sufficiently so that you don't feel it. Moreover, *you never get closer to or further from the center of the earth*, and therefore you don't get the "plummeting" feeling in your stomach (which happens from the sudden removal of the pressure of soft stuff against stiffer stuff in your body which you are used to feeling all the time).

Consider a particular location on Earth—Santa Paula, California, for instance. Since Santa Paula is at S, where arc ES = 35° (about), therefore Santa Paula traces out circle SALB every 24 hrs.

$\angle CSM = \angle SCE = 35°$

so $\angle SCM = 180° - 90° - 35° = 55°$

Calling CS "1", it follows that

$SM = \sin\angle SCM = \sin 55° = .819152044$

So, calling CS "3960 miles" (the radius of Earth)

$SM = (.819152044)(3960) = 3243$ miles.

Hence the circumference of circle SALB is determined:

$SALB = 2\,\pi\, r = 2\,\pi\,(SM) = 2\,\pi\,(3243)$ miles $= 20{,}381$ miles.

Now Santa Paula makes that circle once in 24 hours, uniformly (pretty much).

So Santa Paula goes 20381 miles in 24 hours.

So it goes 849.2 miles in 1 hour.

So it goes 14.2 miles in 1 minute.

Now, *how curved is that 14.2 miles*? Let the 14.2 mile arc be called arc SG.

Well, since the arcs are as the angles, 14.2 miles has to 20831 miles the same ratio that the angle SMG has to 360° (the full circle). Solving, $\angle SMG = .2508°$ (about one quarter of a degree).

Make $\angle GMH$ half of $\angle SMG$, SHG being the chord.

Hence ∠GMH = ½ ∠SMG = .1254°

So ∠MGH = 180° – 90° – .1254° = 89.8746°

Calling MG "1", MH = sin∠MGH = sin(89.8746°) = .999997605

Calling MG "3243 miles," MH = (.999997605)(3243) = 3242.992233 miles.

Now FH = MF – MH = 3243 – 3242.992233 = .007766 miles.

But 1 mile is 5280 feet, so FH = (.007766)(5280) feet = 41 feet.

So, in Santa Paula, in one minute, we move 14.2 miles in arc SFG, but only up to a height of 41 feet above the straight line SG, and back down. Now, suppose you are on a very smoothly operating elevator, and it lifts you up 41 feet and back down in 60 seconds. Will you feel it? No. That means going up 20 feet, smoothly, in 30 seconds. Imagine being on a plane, in clear skies without turbulence. The jet goes up 20 feet in 30 seconds. Do you notice that? No way.

Also, on Earth you are *always* shifting thus—you don't start doing it and then stop and then start again.

And although you effectively weigh slightly less due to the "throwing out" of the earth's motion (as the equator bulges a bit), this is not only slight, but constant, so you never feel your weight shift.

COPERNICUS

DAY 30

THEORETICAL REASONS FOR PREFERRING THE COPERNICAN MODEL, AND A DETERMINATION OF THE DIRECTIONS IN WHICH PLANETS MUST MOVE.

In *De Revolutionibus* 1.10, Copernicus begins to build up his model of the universe.

1. He says everyone is agreed that the heaven of the fixed stars is the highest up, i.e. the furthest away from us, of all the celestial objects.

2. Also, the ancient philosophers are in agreement about the moon being closest, and Saturn the furthest, among the wandering stars. After Saturn, Jupiter is the next closest to us, then Mars is closer still.

3. About Venus and Mercury there is some disagreement.

Ptolemy says "*both* below the sun" because

(a) Venus and Mercury are tied to the sun differently from the way the outer planets are, and therefore it is natural to separate them by the solar sphere. (A bit lame.) Also, because . . .

(b) they fill up the space between Earth and sun so nicely (see p.522, "in order for such a vast space not to remain empty"). That is an argument, apparently, from aesthetic considerations. Ptolemy and his followers say we do not see phases or transits either because Venus and Mercury glow with their own light, or the sun lights them right through, or because "they are small bodies in comparison with the sun, . . . and therefore it would not be easy to see such a little speck in the midst of such beaming light." (p523)

Alpetragius says "*Venus* above, *Mercury* below," but gives

(c) no reason.

Platonists say "*both* above," because

(d) if below, we would see solar transits, but we don't. (This is a feeble argument, however. You don't see them with the naked eye only because the little planet gets swallowed in the light of the sun.) Also, because

(e) if below, Venus and Mercury would exhibit phases (the Platonists assume, correctly, that the planets receive their light from the sun), but they don't seem to do so. (Again, not a very strong argument, at least not today. They do exhibit phases; it is just difficult or impossible to observe them with the naked eye.)

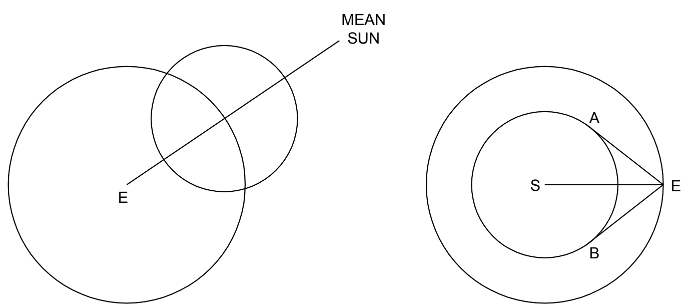
4. Copernicus finds fault with Ptolemy's view. He says that Venus's epicycle, according to Ptolemy, is huge. So what fills up all *that* empty space?

Again, the idea that the sun must be between the "inner" planets and the "outer" ones because this naturally divides those which have a limited elongation from the sun from those which have a limited one, is nonsense—the moon, after all, also has unlimited elongation from the sun. So, then, should we say it is further away from us than the sun, and belongs with the outer planets? (The moon comes between us and the sun, sometimes, remember!)

5. GENERAL ARGUMENT FOR SAYING PLANETS CIRCLE THE SUN.

Based on the foregoing indecision of the geocentrists, Copernicus says we must choose: *either* there is no sure reason why Saturn is furthest out, Jupiter next, and for the order in general, *or* the planets must circle some center other than Earth. Again, *either* we can give no reason why Venus & Mercury have limited elongations from the sun (and so for them S = L), *or* these planets go around the sun.

This, in general, is how he will argue: assume that the planets, including Earth, all go around the sun, and only then can we give a *reason* for the order of the planets, for limited elongations or unlimited ones, for S = L, for S = L + A, for the planet being at perigee when in solar opposition, etc. Otherwise, these are merely additional facts (or assumptions) for which there is no reason, and the universe is largely unintelligible.



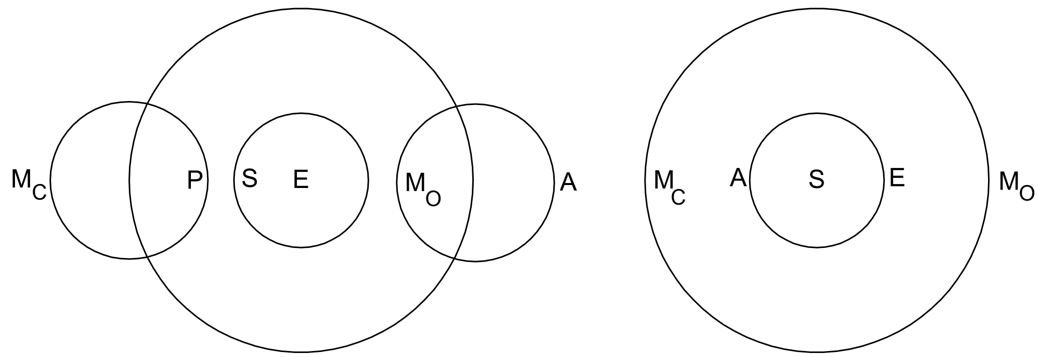
6. VENUS GOES AROUND THE SUN.

Why must the Ptolemaic epicycle of Venus travel with the speed of the mean sun? To Ptolemy, it is just a cosmic coincidence. If we assume Earth is the center, there is no necessity to it. It is an *ad hoc* assumption added into the hypothesis sheerly because the appearances demand it. There can be no *cause* given. Thus Ptolemy.

But for Copernicus, obviously Venus can never appear (from Earth) to be a greater angle from the sun than ∠AES or ∠BES, because it circles the sun as its center, and we do too, and our orbit contains the Venusian one, so the furthest it can get from the sun, in our view, is the angle formed by sun-Earth-Tangent. And since there is a limit to how far Venus can appear from the sun, it will also, on average, travel with the speed of the Mean sun, i.e. S = L.

Since we can give a reason for the truth in one case, but cannot give one in the other, the Copernican hypothesis is more probable. The same works for both inner planets.

Again, on this hypothesis, we can say that Mercury's orbit is inside of Venus's, which is WHY its maximum elongations from the sun are less than those of Venus. No reason can be given for this in Ptolemy. It must be simply assumed as a mere coincidence.



7. MARS GOES AROUND THE SUN.

Ptolemy assumes that Mars is at apogee when it is in solar conjunction, and at perigee when it is in solar opposition.

But *why* must this be so? Why must Mars be at M_c when it is in conjunction with the sun? Nothing in the figure necessitates that. It could just as well be at P, and then it would be at perigee, not apogee. Likewise Mars need not be at M_o when in opposition; it could just as well be at A, putting it at apogee. The reverse is assumed only to match the appearances (presumably). Thus Ptolemy.

For Copernicus, however, it is a necessity just from the geometry of the figure that Mars be at apogee when in solar conjunction (at M_c) and at perigee when in solar opposition (at M_o). Switching the M's obviously makes no difference, and neither would moving the Earth to A make any difference to the rule.

So, since we can give a reason for the facts on the Copernican hypothesis, but not on the Ptolemaic one, the Copernican one is more probably the truth.

The same goes for Jupiter & Saturn.

8. THE *EARTH* ORBITS THE *SUN*.

Copernicus gives a reason for this which assumes that Venus and Mars go round the sun. If he is right in saying that Venus orbits the sun, and so does Mars, and we are outside the sphere of Venus, and the sphere of Mars is outside us, *then we are in the spherical space between these spheres*. If you think of those spheres as hard, crystalline things that cannot interpenetrate, then we are free to move (together with our Moon) only inside that spherical space between—which is around the sun.

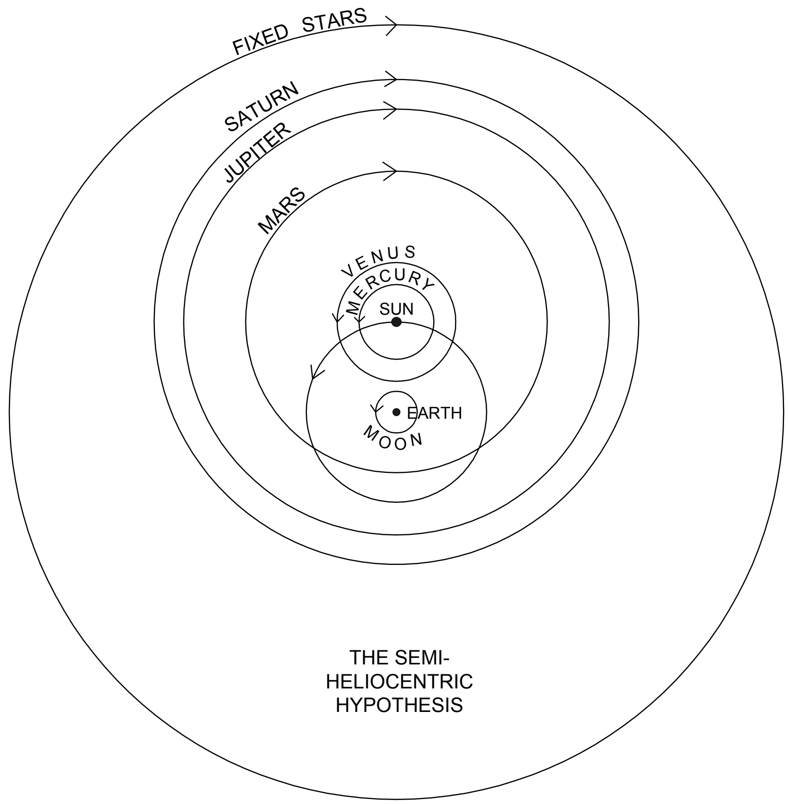
9. THE *SUN* IS IMMOBILE. Copernicus asserts next that the sun does not move: “the center of the world [i.e. universe] is around the sun. I also say that the sun remains forever immobile and that whatever apparent movement belongs to it can be verified of the mobility of the earth.” Whatever movements the sun appears to have are really due to movements that we, the observers on Earth, have. Nevertheless, there is some ambiguity in this—do the planets orbit the physical sun, or the “mean sun”?

10. THE SOLAR SYSTEM IS TINY. Here Copernicus asserts what he must assert, namely that not only Earth, but even all the spheres around the sun, are as a point

compared to the sphere of fixed stars. This is an advance in the understanding of the dimensions of the universe.

And he repeats the idea that his theory is the more convincing because it is simpler—i.e., it uses fewer spheres, and it makes one thing, namely the motion of Earth, the cause of many effects.

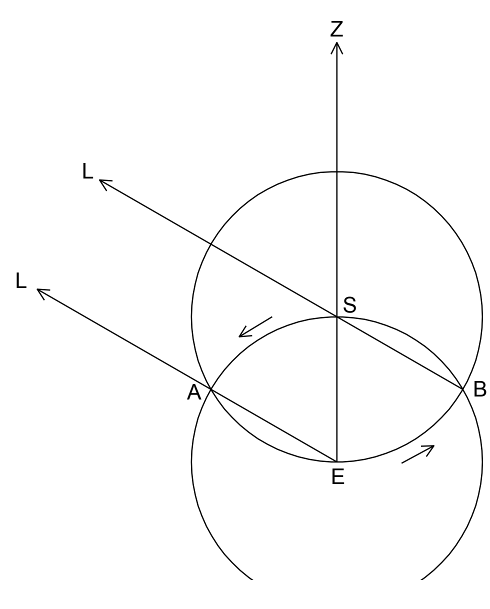
11. IN WHAT DIRECTION DOES THE *EARTH* MOVE? Copernicus assumes that all the planets, including Earth, are going in the same direction around the sun. Assuming Ptolemy's system basically agrees with the appearances, we can see why Copernicus concludes this.



In Ptolemy, all the planets have a basic "eastward" or "antagonistic" or "backward" or "motion of the other" movement through the zodiac. That includes the sun. All these motions, we saw, could be explained by putting the inner planets on same-direction epicycles around the sun as center, and the outer planets on opposite-direction epicycles (or moving eccentrics) about the sun as center, but large enough to contain the earth inside those epicycles. Meanwhile, the moon, too, has a "backward" motion, but it moves faster than the sun (it gets further east of the sun by about 40 minutes a day, i.e. it goes around once in about 30 days).

In the semi-heliocentric model, the earth is immobile, and the sun moves around it, together with everything that moves around the sun. What must we say is happening, then, if we say that the sun is sitting still?

(a) We must say that all the seeming motion of the sun is due to our motion around it. So the sun does not make its circle around us, but we make an *equal* circle around it.

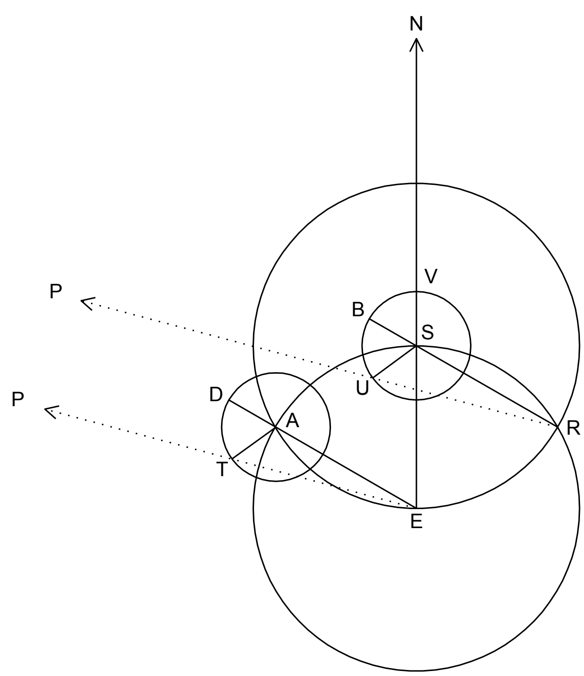


(b) And since the sun appears to move counter-clockwise around us, we must in fact be moving counter-clockwise around it, to keep the Earth-sun line pointing at the same stars. Take any relative position of Earth and sun, at E and S. Then the sun appears to be in location Z in the zodiac. If we let the sun move from S to A around us, then it appears to be at L in the zodiac. The only way to get it to appear in the same place in the zodiac if Earth moves around the sun (at the same speed the sun seems to move around the earth) is if Earth moves from E to B, so that arc EB is equal to arc SA. For then BS will be parallel to EA, and so BS will also point to L (since the distance between the parallels EA and BS is insignificant compared to the distance out to the fixed stars).

12. IN WHAT DIRECTION DO THE INNER PLANETS MOVE?

(a) The INNER PLANETS continue to move counter-clockwise, but their epicycles are now simply their orbits, since the movement of their epicycles on the deferent (i.e. of the sun around Earth) has now been replaced by the movement of Earth around the sun. Suppose E is Earth, S is the sun, and V is Venus on its epicycle, appearing at N in the stars (solar conjunction, just for fun). For Ptolemy, as the sun goes from S to A, we let Venus go from V (i.e. D) to T on its epicycle, so that Venus appears out along line ETP. So for Copernicus, to keep the appearances the same, we must let Earth go from E to R so that arc ER = arc SA. Then we draw RU parallel to ET, and place Venus at U. From the parallels RU and ET we see that Venus will again appear at P in the fixed stars, so that the hypothesis is now equivalent.

(b) But notice the difference! For Ptolemy, Venus traveled only ∠DAT on its epicycle, while in the same time, for Copernicus, it went ∠VSU around the sun. But ∠DAT = ∠BSU (since RS is parallel to EA, and RU to ET). So Copernicus has Venus go ∠BSV *more* around the sun than Ptolemy did, in order to get the same appearances. And, of course, ∠BSV = ∠ESR (vertical), which is the amount of angle the Earth travelled around the sun. **So Earth's motion "undoes" a certain amount of Venus's motion**, because they are in the same direction.



13. IN WHAT DIRECTION DO THE OUTER PLANETS MOVE?

The OUTER PLANETS must *reverse* direction on their opposite-direction epicycles or moving eccentrics, and go in the same direction as the inner planets and Earth around the sun. This is shown by the accompanying diagram as follows. Start off with Mars, sun, Earth all in a straight line (for simplicity).

For PTOLEMY, the sun goes to S, and Mars goes to M, so that the planet appears out at P.

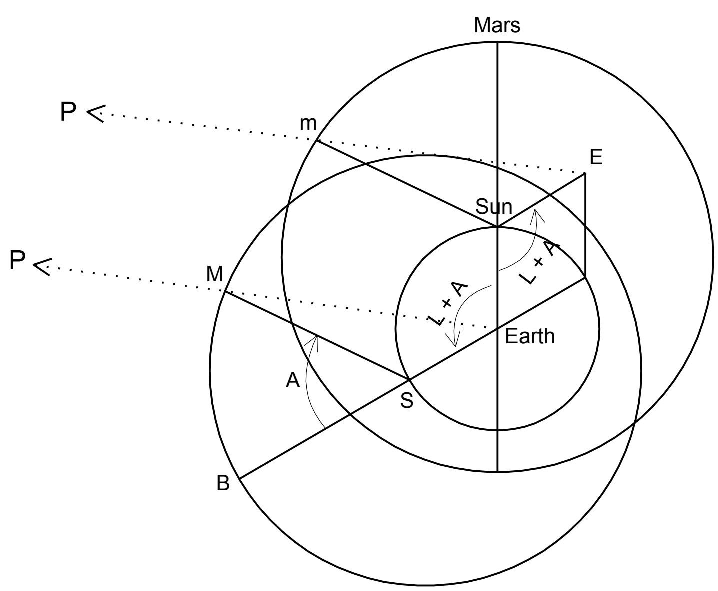
For COPERNICUS, the earth goes to E (just make line “Sun-E” equal and parallel to line “Earth-S”), and so if we draw Em parallel to Ptolemy’s “Earth-M”, Mars will again appear at P in the fixed stars.

So for Ptolemy, Mars moved from “Mars” to M, i.e. from B to M on its giant epicycle. But for Copernicus, to get the same appearances, it moved from “Mars” to *m*, which is the opposite direction of the star on the moving eccentric, but the same direction in which Earth goes around the sun.

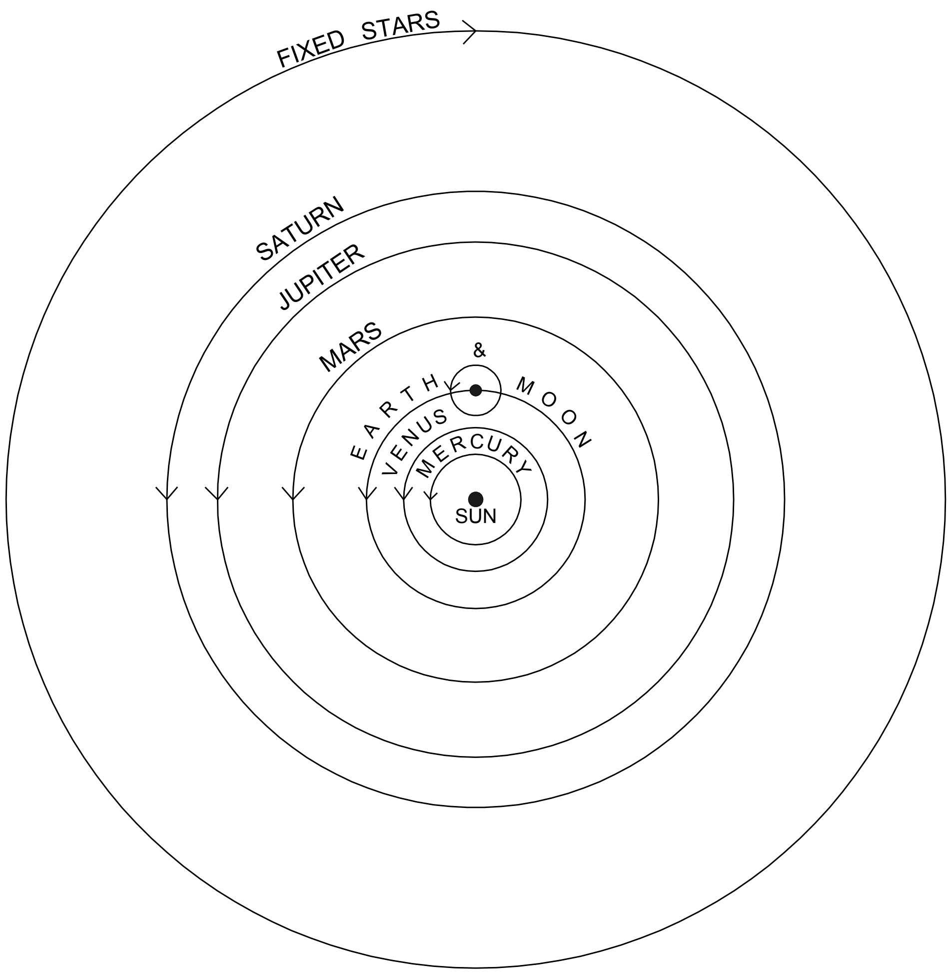
And we have 2 congruent triangles: rm-E-Sun and △M-Earth-S.

And it is evident that ∠ (Mars-Sun-m) = ∠ (Mars-Earth-S) – ∠MSB.

Hence ∠ (Mars-Sun-m) = (L + A) – A = L



14. THEREFORE the diagram we presented earlier correctly depicts the Copernican hypothesis. All the planets, including Earth, move around the sun in the same direction. That simplicity and uniformity and coherence, too, should move us to embrace Copernicus.



THE
HELIOCENTRIC
HYPOTHESIS

COPERNICUS

DAY 31

THE NEED TO REINTERPRET PERIODS; HOW TO EXPLAIN ANOMALIES.

1. The *first law* is still safe, Copernicus now goes on to say. He means by this that, although the heliocentric model forces him to say certain new things, such as "the solar system and planetary spheres have no significant size compared to the outermost sphere of fixed stars," he can still retain the "First Law," namely that *the order of the planets should be determined by how long it takes them to go round once.*

The *sphere of the fixed stars* is furthest out, and, like the sun, it is immobile. So any motion it appears to have is really due to some motion in us.

Saturn is next (30 years)
Jupiter is next (12 years)
Mars is next (2 years)
Earth is next (1 year)
Venus is next (7 ½ months)
Mercury is closest to the sun (88 days)

How did Copernicus get these numbers, though? Compare them to Ptolemy's periods, determined from the least periodic joint returns. "L" is the movement of the planet's epicycle in longitude, "A" is the movement of the star on the epicycle (for the epicyclic hypothesis), "S" is the number of circuits of the sun accomplished in the same time. For an inner planet, S = L, for an outer, S = L + A.

Saturn:	59(S) = 2(L) + 57(A)
Jupiter:	71(S) = 6(L) + 65(A)
Mars:	79(S) = 42(L) + 37(A)
Sun:	1 year to go around once
Venus: 8(S) =	8(L), accomplished at same time as 5(A)
Mercury:	46(S) = 46(L), accomplished at same time as 145(A)

Start with the sun. Since its motion is replaced by ours, ours takes the same time as the sun did for Ptolemy, i.e. one year to go around the sun once. Easy.

Next, take Venus. Since we are going around the sun in the same direction that it is, and Venus *seems* to orbit the sun only 5 times in 8 years (on its "epicycle"), during which we ourselves actually orbit the sun 8 times, that means in those 8 years we have "undone" 8 of Venus's orbits! Therefore it really orbits the sun 5 + 8 times during those 8 years, so it has a period of going around 13 times in 8 years, i.e. one time in 8/13 of a year, i.e. one time in 7 ½ months, as Copernicus has it.

Likewise, Mercury *seems* to orbit the sun 145 times in 46 years, during which time we orbit the sun 46 times, so it really went around 145 + 46 times in 46 years, i.e. 191 times in 46 years, or 1 time in 46/191 years, i.e. in 88 days, as Copernicus has it.

Now, the story is a little different for the *outer planets* as opposed to the *inner*. Why? Because Earth is moving *slower* than the inner planets, and is *outside* their orbits, whereas it is moving *faster* than the outer planets, and is *inside* their orbits. Since we are outside the orbits of the inner planets, we see limits to their elongations, and hence their average speed L around us must appear to be S, that of the mean sun, while their cycles in anomaly A, i.e. around their epicycles, get undone by that same number, S, so to get the true number of times they have gone round in a given time, we have to add A + S.

But with the outer planets, since we are inside their orbits, we do not see any limit to their elongations, and so there is no need for their average speed around us to be S (so L will not equal S). Also, since we are moving *faster* than they are around the sun, we therefore overtake them some number of times during which they go around only once. We "lap" them. And these overtakings will correspond to (and cause) the anomalies, i.e. stations & retrogradations. So "A" will now correspond to the number of times Earth overtakes the outer planet, and L will simply remain the same, i.e. will still represent the number of times the planet goes around, i.e. goes around the sun. So to get the length of time for one cycle in longitude, we just divide the joint-return value of S by the joint-return value of L, e.g. for Saturn divide 59(S) by 2 and you get about 30 years for one cycle of L, as Copernicus has it.

For Jupiter, divide 71(S) by 6 and you get about 12 years for one cycle of L, as Copernicus has it.

For Mars, divide 79(S) by 42 and you get about 2 years for once cycle of L, as Copernicus has it.

Hence the *first law* remains safe, i.e. the order of the spheres going out from the sun corresponds to the order of their speeds. The further from the sun, the slower they go.

2. AN ARGUMENT FROM SUITABILITY. Copernicus says that the sun belongs at the center of the universe, since it is the lamp of the world. It is the lantern; the pilot of the world. Later, Kepler will complain that Copernicus did not take this far enough, since he still makes the Mean sun the center of the movements of things (that absurd fiction at the center, a fiction which gives no light at all!). Kepler will insist on giving the place of honor to a real body, the physical sun.

3. THE COMMENSURABILITY OF THE WORLD.

This we have seen before, with Tycho Brahe. If we adopt Tycho or Copernicus over Ptolemy, and make things go round the sun, then we have a commensurable universe, that is, we can determine the ratios of all distances in the solar system, since the unit Earth-*sun* is a common distance to all, entering into all the triangles.

4. WHY S = L + A.

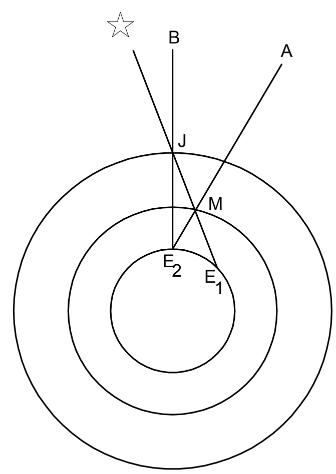
We already saw why S = L, given Copernicus. The inner planets have orbits contained by ours, therefore have limited elongations from the sun, therefore have the same average apparent speed around us that the sun has.

But why does S = L + A for an outer planet?

Because for Copernicus, the cause of "A," i.e. the number of times an outer planet retrogrades, is that the Earth catches up with the outer planet and passes it by (we will see this better below). But of course it must be the case that "The number of times we go around in X time" is equal to "The number of times the planet goes around in X time" PLUS "The number of times *we passed the planet*" (i.e. S = L + A). Think of two race cars on concentric tracks, with the guy on the inside track going faster than the guy on the outside track. Let me be "inside guy" and you can be "outside guy." Then suppose I go around 10 times during the same time that you go around only 4 times. Plainly, then, I lapped you 6 times. So "# of times faster guy goes around" is equal to "# of times slower guy goes around" PLUS "# of times faster guy passed the slower guy."

Since S is really E for Copernicus, i.e. the speed of the earth around the sun, the law is really E = L + A.

5. WHY ANOMALY IS OF GREATER MAGNITUDE IN A NEARER OUTER PLANET THAN IN A FURTHER ONE.

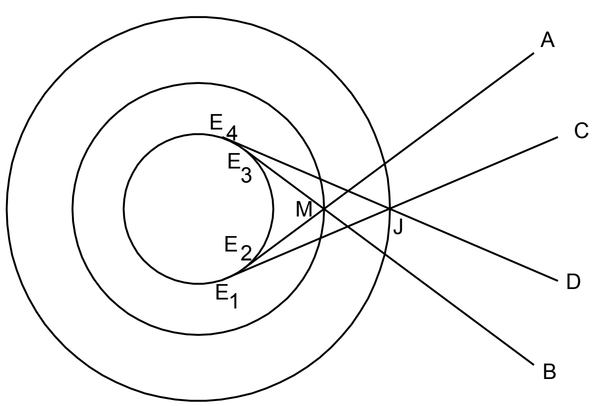


Copernicus first states the fact that the anomaly (i.e. the apparent retrograde motion of an outer planet) is greater for a nearer planet than for a further one, and says his system offers a reason for it.

Why does the retrograde motion of Mars appear greater than that of Jupiter?

When Earth is at E_1 , Mars and Jupiter are roughly in line with some fixed star. Since Earth is on a smaller orbit, it moves faster than Mars and Jupiter—so let's consider them as relatively still while Earth goes from E_1 to E_2. At E_2, Mars will appear to have regressed to A, but Jupiter, only to B.

Or, since retrograde motion is terminated by the stations, and the stations occur when the line joining Earth to the outer planet is tangent to Earth's orbit (we are imagining that Earth is moving while the outer planets sit still, a bit of a fudge; also, this is *not* the Ptolemaic epicycle, things are simpler, so station does occur at tangency), we can show that the regression of Mars is greater than that of Jupiter. For, when the earth



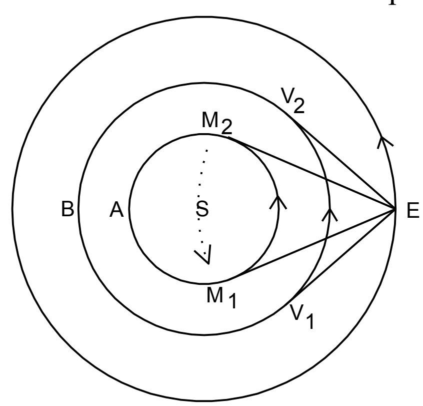
goes from E_1 to E_4 (the tangents from its orbit drawn out to Jupiter), Jupiter appears to retrograde only from C to D. But when the earth goes from E_2 to E_3 (the tangents from its orbit drawn out to Mars), Mars appears to retrograde from A to B.

6. WHY ANOMALY IS OF GREATER MAGNITUDE IN VENUS THAN IN MERCURY.

Since Earth has a larger orbit than Venus & Mercury and contains both, and they move faster than Earth, imagine Earth as relatively still while the two planets buzz around. Retrograde motion is between stations, and stations occur when the line joining Earth to the inner planet is tangent to the planet's orbit, i.e. at M_1 and M_2 and V_1 and V_2. Now since progression is in the direction that the sun appears to move (the motion of the other is counterclockwise to each, like two clocks connected by a single hand, and you rotate one counterclockwise around the other), therefore progression of the planets is $M_2 – A – M_1$ and $V_2 – B – V_1$. Therefore retrograde is represented by the perigeal arcs $M_1 – M_2$ and $V_1 – V_2$, and so it is greater for Venus.

7. WHY ANOMALY OCCURS MORE FREQUENTLY IN A FURTHER OUTER PLANET THAN IN A NEARER ONE (e.g. in Saturn more than in Jupiter), and also why it happens more often in Mercury than in Venus.

That is, retrogradation occurs more often in the more outer of the outer planets, and more often in the more inner of the inner planets.



Why? Because retrogradation happens in the *outer* planets whenever Earth *overtakes* a planet, and it overtakes slower planets more often. But the slower the planet, the more outer it is. Hence retrogradation happens more often in the more outer of the outer planets.

But retrogradation happens in the *inner* planets whenever Earth *is overtaken* by a planet, and it is overtaken more often by the faster planet, Mercury. Hence retrogradation happens more often in Mercury than in Venus.

8. SUMMARY OF COPERNICUS'S EVIDENCE FOR HIS SYSTEM:

- Fewer circles, a simpler model all around.
- Can explain why S = L (and why Venus must have phases).
- Can explain why S = L + A.
- Can explain why an outer planet is at apogee when in solar conjunction, why it is at perigee when in solar opposition.
- Can explain why anomaly occurs in outer planets, i.e. stations etc.
- Can explain why anomaly is greater in Jupiter than in Saturn, and smaller than in Mars.

- Can explain why there is more frequent anomaly in Saturn & Jupiter than in Mars.
- Preserves the “first law”.
- Makes possible a common measure of all distances.
- Makes the earthly thing move, not the divine thing.
- Makes the placed move, not the place.

COPERNICUS

DAY 32

THE MOVEMENTS OF THE EARTH;
HOW TO EXPLAIN STATIONS AND RETROGRADATIONS.

Today we will cover material drawn from *De Revolutionibus* 1.11 and 5.2-4.

Book 1
Chapter 11

1. THE FIRST MOVEMENT OF THE *EARTH*.

Copernicus now begins assigning definite motions to Earth. The first of these, of course, is its axial rotation, from west to east, one time round in about 24 hours.

To get the direction clear, just consider the diagram: It is a matter of fact that when the sun is up in New Hampshire, it is not yet up in the places further west, e.g. California. So the rotation of the Earth brings the eastern parts of it to face the sun before the western parts. So the rotation is of the western parts toward the eastern parts, or counterclockwise looking down at Earth from above the north pole.

2. THE SECOND MOVEMENT OF THE *EARTH*.

The second movement of Earth is its annual orbit around the sun in a plane to which Earth's axis (and equator) is inclined roughly 23½°.

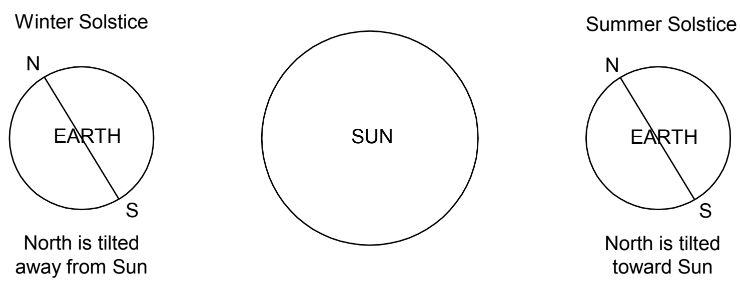
This motion is again from west to east, or counterclockwise looking down at Earth's orbit from the north. Why? Well, the appearances for Ptolemy dictated that the sun move counterclockwise around Earth, as seen looking down from the north (i.e. that the sun move eastward through the fixed stars). And if A makes a circle around B counterclockwise, then (from the same point of view, above) B also makes a circle around A counterclockwise (just imagine two clocks in the same plane, connected by a single hand, and rotate one about the other; if the hand goes clockwise around one, so too around the other).

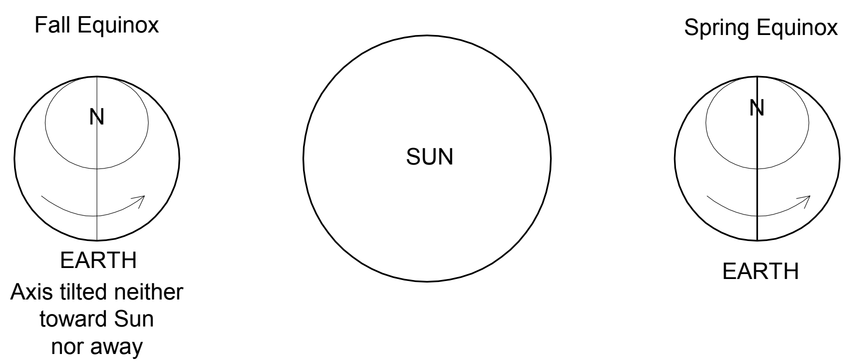
So what is the "ecliptic" now? It is the **projection of the plane of Earth's orbit out to the sphere of fixed stars**.

But how do we explain the equinoxes and the *seasons*? We must assume that Earth's axis remains parallel to itself (almost) as it orbits the sun, since, if it did not, we would have different pole stars throughout the year. But this will also explain the seasons. At one time of year (summer), Earth's North Pole is pointed as directly toward the sun as it can be, and so the north hemisphere gets as hammered by the sun as it can. At the opposite time of year (winter), 180° away on Earth's orbit, it is the South Pole that is pointed maximally toward the sun, and North is pointed as far away as it can be, and its days are shortest. At the EQUINOXES, the tilt of Earth's axis is neither toward nor away from the sun (it's neutral), so every place on Earth gets 12 hours of daylight, 12 hours of darkness (except at the N and S Poles, and near them).

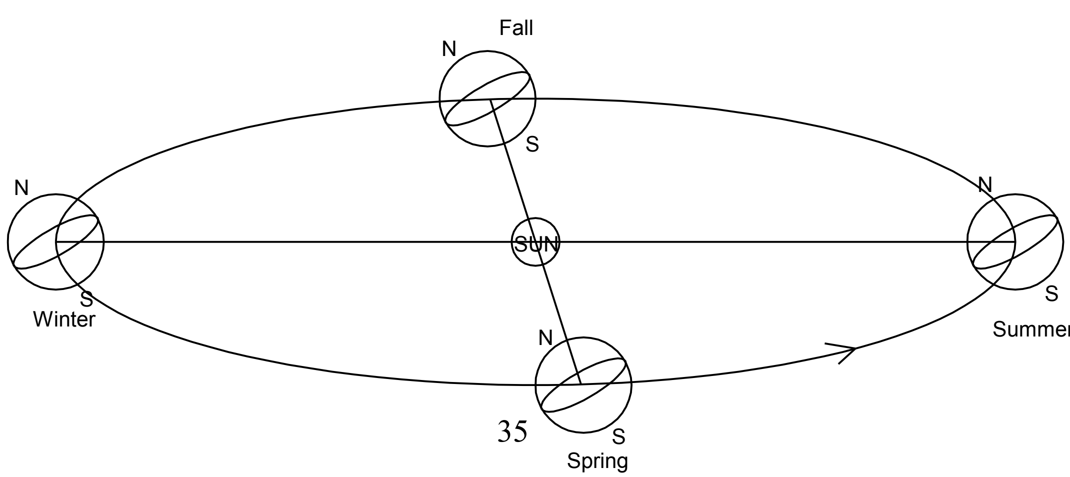
So Earth's axis must be tilted, or there will be no change of seasons.

But it must stay parallel to itself, or there will be different pole stars all year.



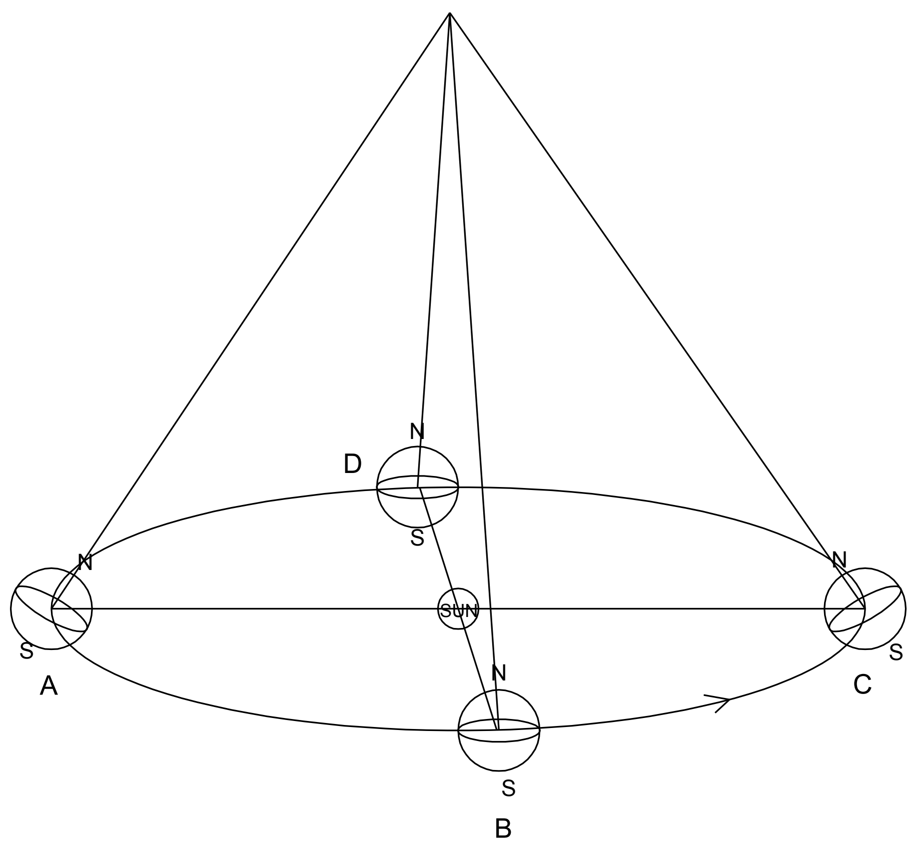


Seasons for northern hemisphere:



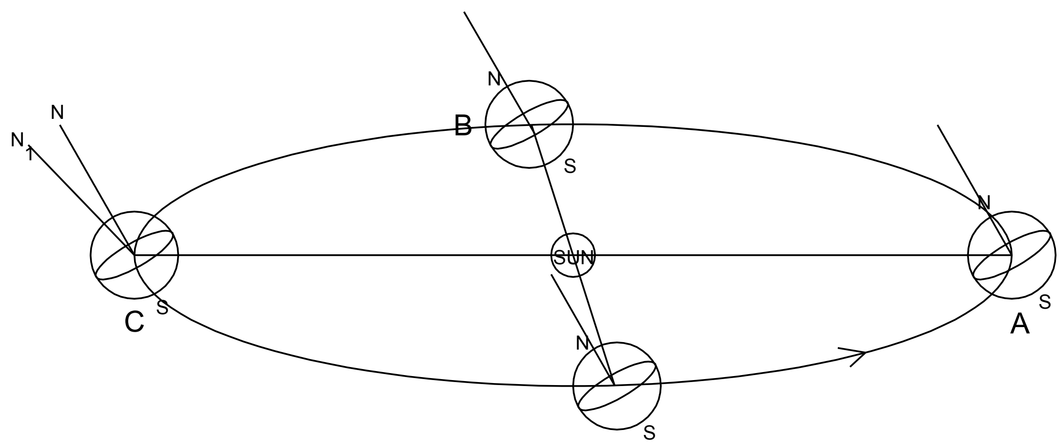
3. THE THIRD MOVEMENT OF THE *EARTH*.

This movement which Copernicus assigns to Earth is a compensatory movement to keep Earth's axis always parallel to itself. According to Copernicus, if Earth were "left to itself," its axis would point toward the center of Earth's orbit, like a toothpick stuck in a piece of gum stuck to the rim of a record. So, to stay always parallel to itself, since the axis at C is 180° of rotation off from the parallel to the axis at A, the axis must rotate at the same angular velocity as Earth around the sun, but in the opposite direction.



FOURTH MOVEMENT OF *EARTH?*

In the Copernican model there must also be a way to account for the precession of the equinoxes, which, in the Ptolemaic model, was a slow spin of the celestial sphere about the poles of the ecliptic. The precession of the equinoxes is really due to the fact that Earth's axis wobbles and does not remain perfectly parallel to itself—or, to put it Copernicus's way, the compensatory motion of the axis is slightly faster than Earth's orbital motion—i.e., at C, our North Pole points to N_1 vs. N.



De Revolutionibus, 5.2-3

5. We will now touch on how to deal with the anomalies, that is, the apparent speeding up, slowing down, stations, and retrograde motions. Remember, Copernicus is still unwilling to accept that the heavenly bodies really do in fact speed up and slow down (and he is quite right to suppose that the apparent irregularities, such as stations and retrogradations, are due to our own movement, and not to these appearances simply being real movements in the planets).

Chapter 2

6. Copernicus here complains about the equant and hints that he will eliminate it. "But it is clear that the regularity of the epicycle should occur in relation to E, the centre of its deferent . . . Accordingly they [the ancients] concede that in this case the regularity of the circular movement can occur with respect to a foreign and not the proper center. . . . But I think I have already made a sufficient refutation of that in the case of the moon." Here he refers to his lunar theory, which we have skipped over.

Chapter 3

7. Copernicus will now explain the apparent irregularities of the planets by the effects which the Earth's motion has on the appearances.

He explains that he will keep things simple, for now. He will assume that the orbit of the planet is concentric with Earth's orbit (which he puts around the **mean** sun—since

Earth is on a perfect circle, going perfectly uniformly around it, the center of that circle is now the mean sun; and the physical sun is not exactly there. Shouldn't he be annoyed by this?—A door for Kepler). Also, the orbits of the planets are not all in one plane, although they are close enough that for now we will let them be in one plane.

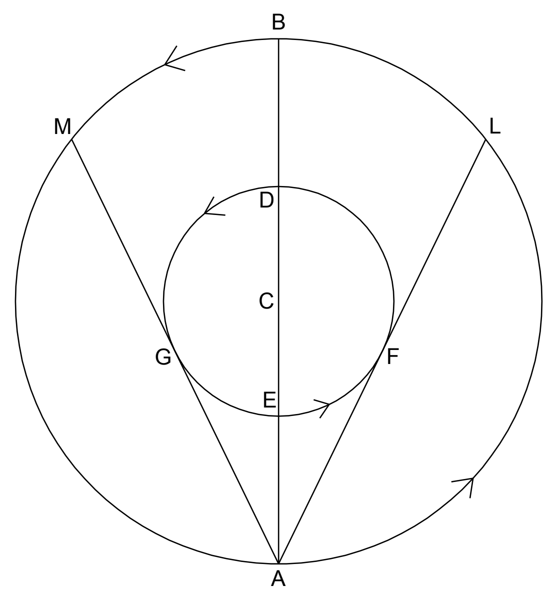
Specifically, we will be explaining various claims which Copernicus makes concerning when stations and retrogradations occur for inner and outer planets, and why they occur.

8. STATIONS & RETROGRADATIONS FOR INNER PLANETS.

Here we are going to explain station and retrogradation in the case of an *inner planet*. That is, we will see how the Copernican system makes it possible to explain why, for an inner planet, retrogradation must occur *at inferior solar conjunction.*

[A] "C and line ACB will appear to the eye borne along at A to move in accordance with the mean movement of the sun."

We are letting ALBM be Earth's orbit, C its center, so C is our "mean sun."



We are letting EFDG be the orbit of an inner planet, concentric with C.

A is Earth, and we draw AG, AF tangent to the inner planet's orbit.

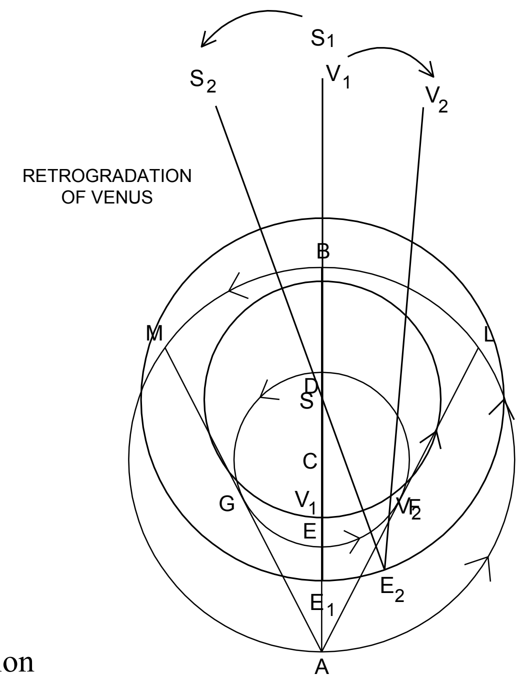
[B] "The planet in circle DFG as in an epicycle will traverse arc FDG eastward in greater time than it will the remaining arc GEF westward."

Why is this true? Because arc FDG is a good deal greater than arc GEF, since GEF is the arc caught between the tangents from A, which must always be less than a semicircle in any such figure. Even if A (Earth) is in motion, this motion is "undoing" the motion through arc FDG as much as it is "undoing" the motion through arc GEF, so we need only consider the difference of magnitude in the arcs—the greater arc will take the greater time.

[C] "In the upper arc" FDG "it will add the total angle FAG to the mean movement of the sun," i.e. in that arc the motion of the planet will be *in addition to* the speed it seems to have just because of Earth's speed around C. If Venus stood still at D, it would still seem to progress just because of Earth's motion. As it is, Venus is also moving, so its motion adds to the progression there (so we get greatest passage at apogee).

But in the lower arc, GEF, which is where regression occurs (because it is on the near arc between the tangents), its motion is subtracted from the apparent progression due to Earth's motion—and since it there exceeds that apparent progressive motion, it appears to retrograde.

9. Let E_1 and E_2 be two successive locations of Earth while Venus is in its perigeal arc (between the tangents to its orbit drawn from Earth), and let V_1 and V_2 be the two corresponding locations of Venus. Then the projected appearances of Venus on the backdrop of fixed stars will make it appear to retrograde.

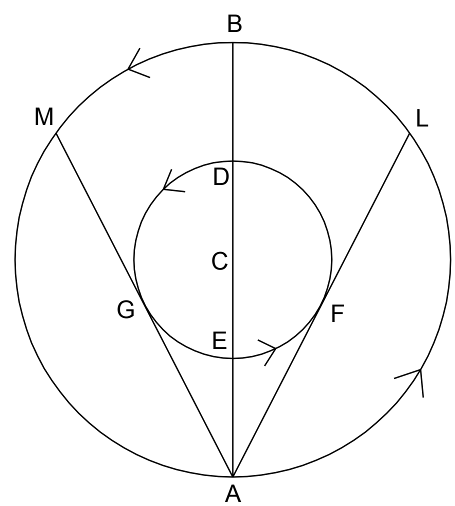


10. [D] "Where the additive movement is equal to the subtractive, the planet will seem to come to a stop."

And this, Copernicus says, will happen at the tangents G and F, where the planet is (for an instant in time) coming straight at us, or going straight away from us.

QUESTION: Why does this not conflict with the Ptolemaic (or Apollonian) rule for finding the station points by *secants*? Actually, it does! As long as Earth is moving with the speed of the mean sun around C, that motion will affect the appearances of the inner planet; at tangency, when the inner planet's own motion contributes nothing to its apparent motion, it will not be at station so long as our own movement makes the planet appear to move. So station cannot occur at tangency, but must occur at the secant (as explained by Ptolemy), where the two motions cancel in the appearances. Without saying so, Copernicus is oversimplifying and abstracting from the motion of Earth, i.e., treating it as though it sat still compared to the inner planet, which moves faster. That is equivalent to Ptolemy letting the epicycle sit still on the deferent—in which case the planet would appear to be at station when it is along the tangent (to the epicycle) which passes through our eye. Similarly Copernicus will be abstracting from the motion of the outer planets when considering their stations, as though Earth (which moves faster than them) alone were moving, while they sat still.

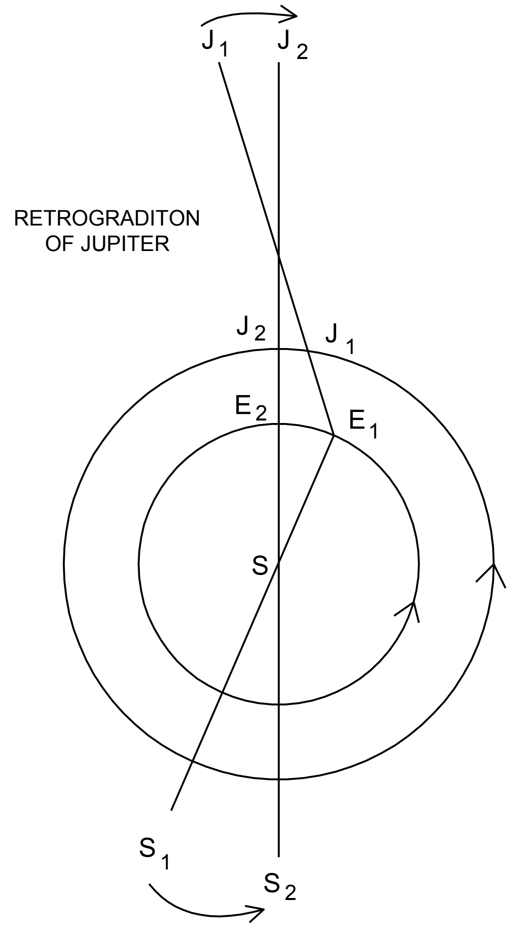
11. Copernicus notes the zodiacal anomaly, namely the fact that the sums of greatest elongations from the mean sun are not everywhere the same (e.g. for Venus), when seen in different parts of the zodiac. He suggests that this can be explained by eccentric circles: "But the greatest angular elongations from the mean movement of the sun, which these planets have in the morning and evening and which are understood by the angles *FAE* and *GAE*, are not everywhere equal, neither the one to the other, nor are the sums of the two equal; for the apparent reason that the route of these planets is not along circles homocentric with the



terrestrial circle but along certain others, by which they effect the second irregularity."

12. Station and Retrogradation for an OUTER PLANET is next.

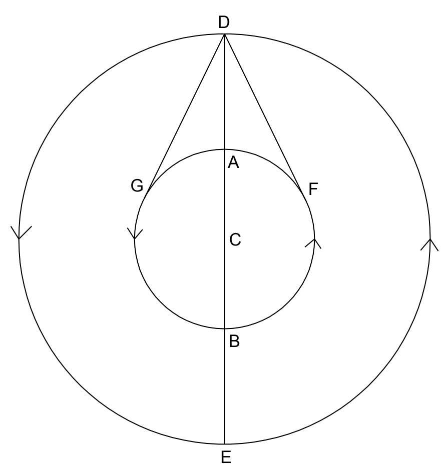
[E] The same sort of explanation accounts for the retrogradations of an outer planet, such as Jupiter. Let E_1 and E_2 be two locations of Earth on its orbit when Jupiter is nearing solar opposition (mean sun is at S, center of Earth's orbit). Let J_1 and J_2 be the corresponding locations of Jupiter at the same times. Since Earth moves faster, its arc will be greater, and let the drawing reflect this. Join the lines from Earth's locations to the corresponding locations of Jupiter, and you will see that the projected locations of Jupiter in the sky will make it appear to retrograde, or go westward, the opposite of the direction of the mean sun (eastward).



So now we can explain why, for an outer planet, retrogradation takes place near solar opposition.

13. COPERNICUS'S DIAGRAM.

Copernicus notes that "From point A only will the true position of the planet ... be apparent". That is, only when Earth, planet, and center of orbits all make one straight line, will the planet's apparent position in the zodiac be its true heliocentric longitude. Something like this was true for Ptolemy, too; although he put Earth at rest and at the center of the universe, he did not make any planet, or even the sun, move around Earth as the center of uniform motion, nor even the geometric center of motion (e.g. for an eccentric deferent).



14. [F] As we said in [C] above for an inner planet, here we say that on arc GBF Earth's movement is in addition to the planet's motion, and in

that time it adds ∠GDF to the motion of the planet itself. To see this, think of yourself first at G, then moving to B, then to F while the outer planet, sluggish compared to Earth, basically stays at D. So the total shift in the planet as seen through that time corresponds to its being seen from G and then from F. How will that affect the apparent position in the stars? Just project those lines GD and FD through D onto the stars, and you will make an angle equal to ∠GDF, vertically opposite to it. So that is the total apparent shift in the planet due to the movement of Earth from G, through B, all the way over to F. Meanwhile, the planet itself has also moved (a little bit), and in the same direction (cooperatively), and hence its motion is *added* to its apparent motion due to Earth's movement.

15. [G] The outer planet retrogrades when Earth moves along arc FAG, since Earth's motion is faster than that of the planet.

COPERNICUS

DAY 33

REJECTING THE EQUANT

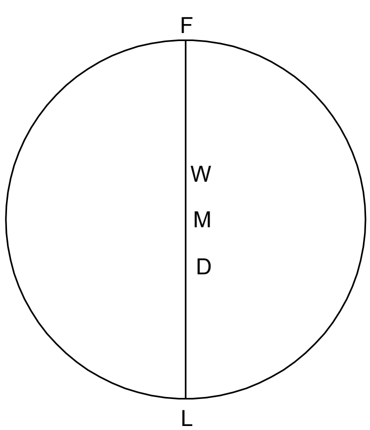
Today we will be covering material drawn from *De Revolutionibus*, 5.4.

Chapter 4

1. SOME VOCABULARY. *Apsis* means “point,” like that in a pointed arch. So *apsides* is the plural form, and the line of apsides is the line through the “highest” and “lowest” apsides or points, namely the aphelion and perihelion. Also, from now on we will talk more about “aphelion” and “perihelion” than about “apogee” and “perigee” (except in lunar theory)!

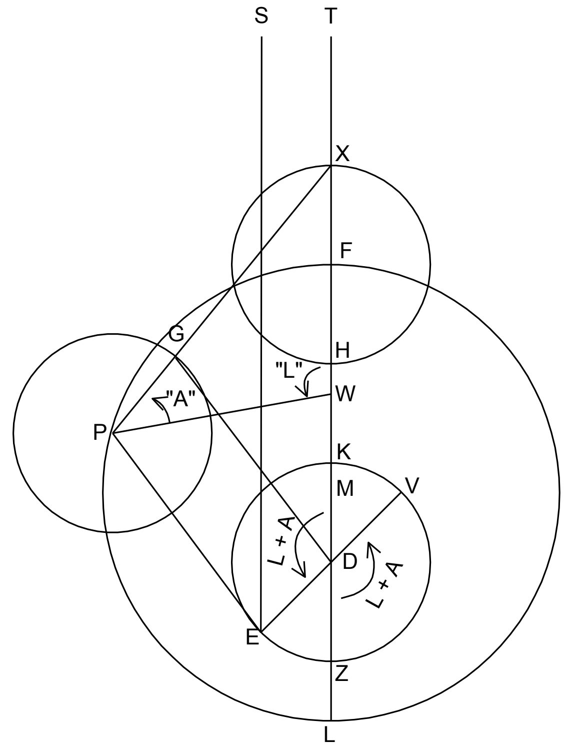
2. REJECTING THE EQUANT.

So far, Copernicus has explained only the heliacal anomaly of the planets—that is, why they have stations, retrograde, have greatest elongations from the mean sun, and the like. Now he wants to account for the zodiacal anomaly, that is, for the fact that the magnitude of these anomalies in each planet seems to depend on where they occur in the zodiac. And he wants to do this without having recourse to the “equant” circle, or equant center—the center about which the planet has uniform angular velocity, but which is *not* the center of its deferent. ***But first let us see what he is giving up, what he could have had*** if only he were willing to put up with an equant:



Imagine a circle FL, center M, mean sun at D, point W such that WM = MD. If we let the planet move on that circle, but such that it sweeps out equal angles in equal times around point W, we get perfect equivalence to Ptolemy. No epicycles at all—the path of the planet itself is simply a circle! (After all, for an inner planet, stations and regressions need no longer be explained by an epicycle, but simply by *a circle which is equal to the former* ***deferent****,* namely Earth’s orbit.)

3. EQUIVALENCE.
Let us see now how this simple hypothesis is, as asserted, equivalent to Ptolemy's (for an *outer* planet):



In the adjacent figure, the epicycle is equal to a circle of diameter EDV, which can be either the orbit of the mean sun around Earth (for quasi-Copernican), or vice versa (for Ptolemaic).

We also draw in ES parallel to DT.

PTOLEMY:

Earth = D, sun goes (from Z to V) = L + A
Epicycle's center goes L about W, i.e. from F to P
Planet goes A about P (or F), i.e. from H to G
Therefore apparent motion is $\angle HDG$.
Produce VD to E, PG to X.

I say $\angle HDG = \angle SEP$

For DE is parallel to PG [since $\angle MDE = L + A$, & $\angle PXT = L + A$ (1.32 in $\triangle PXW$)]
and <u>DE = PG</u> [we made the epicycle = sun's orbit; Ptolemy permits any scale]
so DG is parallel to EP
But <u>HD is parallel to SE</u> [construction]
so $\angle HDG = \angle SEP$
Therefore the apparent motion is equal to $\angle SEP$

QUASI-COPERNICUS:

Letting W remain the equant, but making the inner circle (still equal to the epicycle) now the orbit of Earth around the Mean sun at D, Earth goes (from K to E) = L + A. Since we give this circular motion to Earth, we need not give it to the planet, so we leave out the epicycle in this hypothesis, and let the planet go L about W simply from F to P **(so its path is just the deferent!)**:

Therefore it is seen along KX first, along EP later
But if ES is parallel to KT, the star S and star T are the same,
since Earth's orbit is a point to the heavens.
Therefore the apparent motion is $\angle$**SEP**

And that gives us the equivalence.

NOTE: An outer planet's *epicycle* gets replaced by Earth's *orbit*.

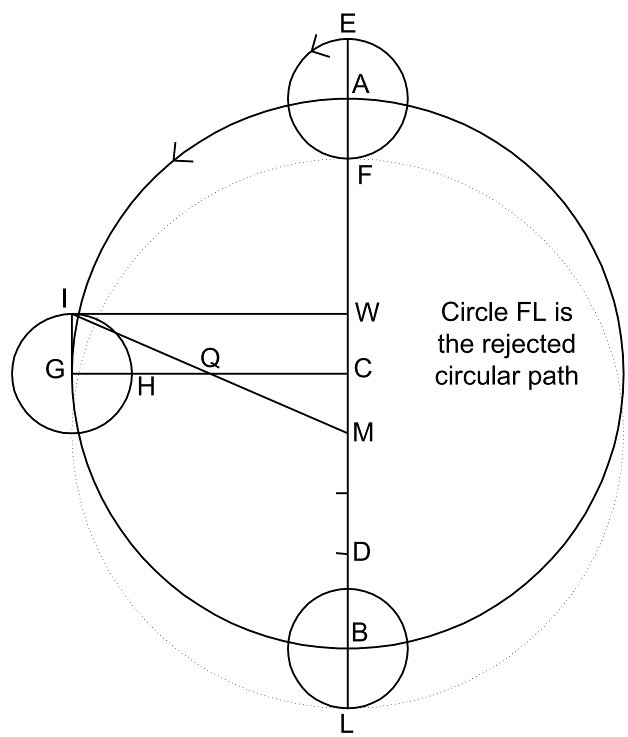
We see that an inner planet's *deferent* gets replaced by Earth's *orbit*.

An inner planet's epicycle is really its own orbit around the sun, and the deferent is the sun's orbit around us, or really our orbit around it.

An outer planet's epicycle is really just an explanation for its apparent anomaly or retrogradation, etc., which is really caused by our lapping it, so that its epicycle is really just our orbit around the sun.

4. WHAT COPERNICUS PREFERS. But instead, because Copernicus hates the equant so much, he opts for a simple epicycle to explain the zodiacal anomaly, and gives up the chance to have the planet's actual *path* be a *circle*! Here is his hypothesis:

AB is the deferent, center C.
It is a same-direction epicycle, starting at A.
Speed on the epicycle = speed on the deferent.
Uniform speed of epicycle is around C.
Uniform speed of star is around the center of the epicycle.
Mean sun is at D.
Cut off CM = 1/3 CD [so MD = 2CM]
Make the radius of the epicycle =AF = CM.



How do we know that the resulting path is NOT a circle? Because by the following considerations we see that the planet's path bulges out at the sides.

The planet starts at F (aphelion), goes to I after **one quadrant**, and ends at L (perihelion).

Therefore the path (not depicted here!) goes through F, I, L.
And FL is a line of symmetry (the planet does the same thing on both sides of it), and therefore *if* the path were a circle, *then* FL would be a diameter.

Now FM = FC + CM = FC + AF = AC = CG
and <u>ML = MB + BL = MB + CM = CB = CG</u>
so FM = ML

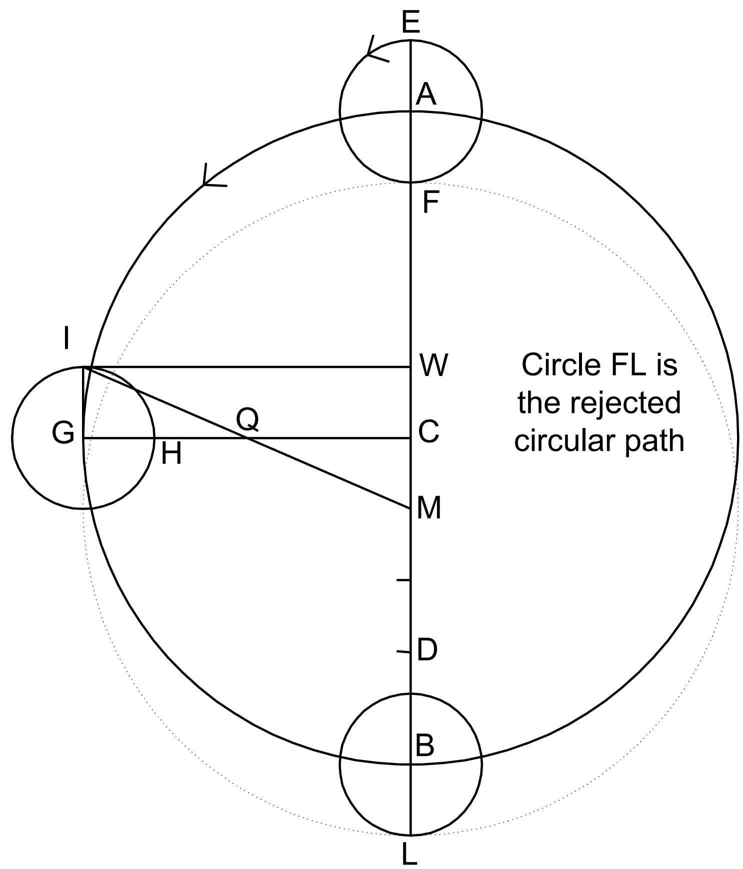
So: *if* the path is a circle, *then* M is the center!
So if the planet's path is a circle, it is the dotted circle on diameter FL, which equals deferent AB, just shifted down.

But QI > QG [hypotenuse QI]
and <u>QM > QC</u> [hypotenuse QM]

so IM > CG [sums]
so IM > MF [FM = CG, just shown above]
so IM in fact reaches *outside* the dotted circle on diameter FL.

Therefore Copernicus's path for the planet is *not* a circle, but bulges out at the sides.

Q.E.D.



5. NOT EQUIVALENT. This hypothesis of Copernicus is not equivalent to Ptolemy's, but when he uses the right proportions for each planet, it differs from Ptolemy's so little in terms of the predicted appearances, that naked-eye astronomy cannot (or cannot easily) distinguish which one is more true to the appearances. The kinds of observations and instruments made in the time of Copernicus would not have enabled anyone to decide.

But Tycho came along and made it possible to tell (still with the naked eye, but aided by better instruments and techniques). Copernicus has the planet bulging out at the sides, but really it should be “sucked in” at the sides. Ptolemy, too, whose hypothesis is equivalent to the dotted circle, has the planet out too far at the sides. But this circle is simple, and very close to being correct, so Kepler keeps coming back to it as a touch-stone for the orbit of Mars. But, as we shall see, Kepler will also replace D, the Mean sun, with the physical body of the sun.

KEPLER

DAY 34

NOTE ON THE LIFE AND WORK OF KEPLER (1571-1630)

Johannes Kepler was born on December 27, 1571 in Weil in the Duchy of Wurttemberg. His family was noble but had fallen on hard times. His father ran a tavern, and young Kepler worked in the tavern and even labored in the fields. In 1584 he became a charity student at the Protestant seminary in Adelberg, and in 1586 he attended the college at Maulbronn. There he wrote a brilliant exam for the bachelor's degree which enabled him to enter the University of Tübingen in 1588. As part of the normal course of studies for his master's degree in philosophy he studied astronomy under Mästlin, who first introduced him to the theories of Copernicus. Kepler wrote a paper reconciling the Copernican model with Sacred Scripture—unlike his later mentor, Tycho Brahe, Kepler was a whole-hearted Copernican. Still, he wished to enter the ministry, and only reluctantly accepted the chair of astronomy at the Lutheran school of Graz when it was offered to him.

In 1596, Kepler published his first astronomical work, *Precursor of Cosmographic Dissertations or the Cosmographic Mystery*. Its main thesis was that the ratios of the orbital radii of the planets were determined by inscription and circumscription of their orbits variously around the five Platonic solids. It seemed to Kepler that God must have had these beautiful objects in view when deciding at what distances to separate them in their orbits. This interest in mathematical patterns among the orbital radii of the planets, and also among their periods, stayed with Kepler throughout his career, and the greatest fruits of that interest are Kepler's Second and Third Law of planetary motion. This first published work of his won him fame and also correspondence with the two most famous astronomers of the time—Tycho Brahe and Galileo.

When the Catholic archduke of Styria issued an edict banishing Protestant preachers and professors in 1598, Kepler fled to the Hungarian border. With the help of some Jesuits, he was reinstated in his original post, but nevertheless gladly accepted Tycho Brahe's invitation to come join him as his assistant at the observatory near Prague. When Tycho died, Kepler was appointed imperial astronomer, and "inherited" Tycho's precious data. Part of keeping on good terms with the emperor meant playing the astrologer. So Kepler wrote *On the More Certain Foundations of Astrology* (1602). Kepler was not the last of the astronomers to be mixed up in astrology—many years later Cassini, too, began his career as an astrologer, although he eventually renounced it altogether and condemned it. Kepler's main distinction is that he was the first true astrophysicist. Ptolemy (and his followers) spent very little time thinking about what made the heavens move as they do, and even less time arguing in defense or this or that understanding of the motive causes in the heavens. Copernicus thought that motion in a circle was natural to a spherical body, and that was enough to satisfy him as a physical account of the motions of the heavens. So he was not much different from Ptolemy on that score. But Kepler was much influenced by Gilbert's book on magnets. He believed that the heavenly motions could be understood as the results of magnetic attractions or repulsions. He was mistaken about this, but this conception of his nonetheless set him apart from his predecessors. Rather than assume the heavens had a completely different nature from the familiar bodies on Earth, he endeavored to understand their motions in light of physical tendencies we find "down here." About a century later, Newton would continue that same project, but show that the motions of the heavens were due not to magnetic attractions, but to another familiar force—heaviness, or gravity.

In 1609 Kepler published the book with which we will be primarily concerned:

ASTRONOMIA NOVA
ΑΙΤΙΟΛΟΓΗΤΟΣ
Seu
PHYSICA COELESTIS
tradita commentariis
DE MOTIBUS STELLAE
MARTIS

Ex observationibus G. V. TYCHONIS BRAHE:

Jussu & sumptibus

RUDOLPHI II
Romanorum Imperatoris &c:

Plurium annorum pertinaci studio elaborata Pragae
A S[a]. C[a]. M.[tis] S[a]. Mathematico
JOANNE KEPLERO

This grand title page can be translated as follows:

New Aetiological Astronomy
or
Celestial Physics
together with Commentaries on the Movements of the Planet Mars

From the Observations of the Gentleman Tycho Brahe,
By the Order and Generosities of Rudolph II, Emperor of the Romans, etc., Worked out at Prague in a Tenacious Study of Many Years,
by His Holy Imperial Majesty's Mathematician Johannes Kepler

The word “Aetiological” means “based on causes,” that is, on physical causes, and particularly on familiar ones, such as magnetic forces. That is part of what is new in Kepler’s *Astronomia Nova*. But there are other new things. There is a new stress upon the importance of minute precision in the observations which are to be the foundation of astronomical theory. There is also a new willingness to start from scratch—to abandon the preconceived notion that the heavenly motions are produced by circular movements. Magnets, for example, do not act “circularly.” And there is also a new interest in the *shape of the orbit itself*, the very path of a planet, and not only in the shape of the supposed mechanical causes producing its motion. In Ptolemy, who placed a star on an epicycle, the actual path of a planet might well have been a curlicue. In Copernicus, as we have seen, the actual path of a planet might be some horrible almost-circle that was a bit pudgy on the sides. He had no desire that the orbital path itself should be intelligible or simple. Moreover, there was no particular interest in discovering exactly what the path was. There was only the old desire to keep the path *a product of perfectly circular motions*. But since Brahe had proved that there are no crystalline orbs out there (thanks to the extremely eccentric paths of the comets which he had shown to pass through, unhindered, several of the supposed “spheres” of the planets), there was little reason to suppose the heavens worked on the basis of circular motions. At any rate, the supposed physical cause of this, namely hard crystalline spheres, had been proven not to exist. This paved the way for a new physics of the heavens, for a new astronomy based on new physics—in other words, for Kepler.

Kepler’s personal life was always difficult. His mother had been tried for witchcraft. The emperor was not very good about paying his salary. His first wife appears to have been clinically depressed. She died, and his three children by her also died (of smallpox). After 1611, Kepler accepted an invitation to be mathematician for Upper Austria, while still retaining his position as court astronomer. He moved to Linz, and remarried in 1613, but he remained poor. Nonetheless, he was able to publish new astronomical tables, the *Rudolphine Tables*, compiled from his own and Tycho’s observations, in 1627. In 1628, he moved to Sagan in Silesia, and Duke Wallenstein of Friedland agreed to have the Emperor’s debts to Kepler transferred to him. But the Duke was only slightly better at paying Kepler than the Emperor had been. In a visit to Ratisbon in 1630, where Kepler hoped to present his financial case in court and get payment on the debts Duke Wallenstein owed him, he got a fever and died on November 15.

There is something else worth noting about Kepler—he was a very colorful, imaginative, and passionate writer. Here is a little sample, drawn from his note “To the Reader” at the outset of Book Four of his *Epitome of Copernican Astronomy*:

> It has been ten years since I published my *Commentaries on the Movements of the Planet Mars*. As only a few copies of the book were printed, and as it had (so to speak) hidden the teaching about celestial causes in thickets of calculations and the rest of the astronomical apparatus, and since the more delicate readers were frightened away by the price of the book too; it seemed to my friends that I should be doing right and fulfilling my responsibilities, if I should write an epitome, wherein a summary of both the physical and astronomical teaching concerning the heavens would be set forth in plain and simple speech and with the boredom of the demonstrations alleviated.

KEPLER

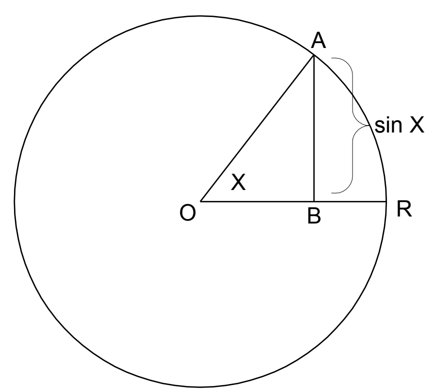
DAY 35

AN UPDATE ON TRIGONOMETRY

In preparation for the study of Kepler, we will here review certain elementary principles of trigonometry which he will use. In our study of Ptolemy, we saw how he developed a table of chords and arcs by means of which to solve for the remaining sides and angles of triangles given certain sides or angles. These days, however, we do not generally do this by means of "chords" but by means of "sines" (and "cosines" and "tangents" and so on). So rather than a "table of chords" we would have a "table of sines" (or a calculator ready to present the entries on that table at the touch of a button!). Also, we do not chop the diameter of our reference circle into 120 parts, but instead we call its radius "1." Again, there are certain basic trigonometric identities and relationships which are extremely useful for solving triangles, and which Ptolemy did not present to us. And Kepler will presume familiarity with all these things. Accordingly, we will now present the basics of modern trigonometry in order to make Kepler's work more intelligible.

1. THE DEFINITION OF "SINE"

Like Ptolemy's table of chords and arcs, a table of sines associates an angle with a unique line length. Also like Ptolemy's table, a table of sines places angles at the center of a circle. To find the sine of an angle, *x* (let it be acute), we draw a circle with radius OR = 1 (as opposed to a radius of 60 in Ptolemy). Place the vertex of angle *x* at O, with radius OR as one of its legs, and from the end of the other leg, point A, we drop AB perpendicular to OR. The *length* of this perpendicular is called the "sine" of angle *x*, written



AB = sin *x*

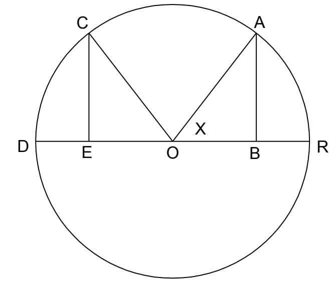
The name "sine" has an odd history. There was a Sanskrit word for "half-chord" (*jya-ardha*, sometimes shortened to *jiva*) which the Arabs translated into their own language (as *jiba*), and an abbreviation for this (it was written *jb*, without vowels) was mistaken by Latin translators for the Arabic word *jaib* for "fold" in clothing, or "bosom" or "bay." So the Latin translators named the half-chord a "sinus," which is Latin for "bay" or "fold in a toga." Not very helpful. AB is not a "bay" or a "fold." It might be more useful to associate the word "sine" with "sinew", since the sine is like half the string of a hunter's bow.

2. HOW SINES AND ANGLES MATCH UP

A given angle, *x*, obviously has only one sine, the length of AB. We cannot get any other length for the side of right triangle ABO which is opposite *x* , so long as *x* is given, and the length of AO is a standard unit.

But does a given sine go with just one angle, *x* ?

Not quite. If we keep opening angle *x* up to a point C so that AC is parallel to diameter DR, then this new angle, ∠COR, has a sine CE which is equal to AB, the sine of *x*.



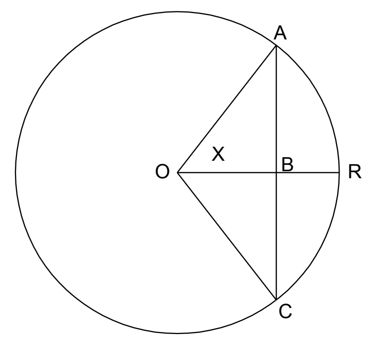
∠COR is the supplement of ∠*x*, since it is plainly equal to ∠AOD.

So sin *x* = sin (supplement of *x*) = sin (180° – *x*).

And there will also be two sines *below* diameter DR which are equal to AB, if we continue to open the angle past 180°. But we need not concern ourselves with that now.

3. SINES AND CHORDS

How do sines relate to chords? Well, if we extend AB down to C, obviously



AC = 2AB

and ∠AOC = 2 *x*

But AC is the chord of ∠AOC.
So, given that a chord AC goes with an angle AOC, we also know that

½ (AC) = sin (½ ∠AOC)

Does that enable us to use Ptolemy's table of chords and arcs as a table of sines and angles? Almost—one more step is needed. We use the same "degree" system that Ptolemy did (although we express parts of degrees decimally, not sexagesimally), so we only need to cut his angle values in half. But we not only have to divide his chords in half—for since Ptolemy's radius is called "60," and the table of sines (by another convention) requires us to call the radius "1," we must also divide the value of his chord by 60 to translate it into "sine language." So we divide his chord-value by 120 in all, and divide his arc value by 2:

1/120 (Ptolemy's chord) = sin (1/2 the corresponding arc)

EXAMPLE:

On Ptolemy's table, a chord of 60 parts goes with an arc of 60°.

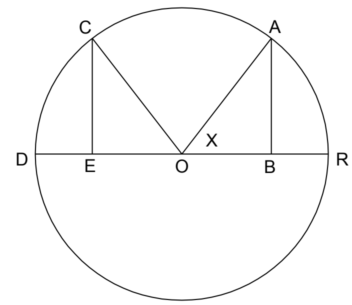
So: 1/120 (60) = sin (1/2 [60°])

i.e. ½ = sin 30°

4. THE DEFINITION OF "COSINE"

We said that perpendicular AB is the "sine" of $\angle x$. But the segment of the radius which it cuts off, namely OB, is also distinctive of $\angle x$. This segment is called the "cosine" of x, written

OB = cos x



Now, does the supplement of x have the same cosine? You may recall from high school that if O is the origin of a coordinate axes system, we read lengths by beginning from it, and those going out to the right of it are (by convention) "positive," while those to the left are "negative."

So if $\angle$COR = $\angle$AOD = the supplement of $\angle x$
then cos ($\angle$COR) = OE
And although OE = OB, if OB is ".64", then OE is "– .64"

5. OTHER TRIG FUNCTIONS

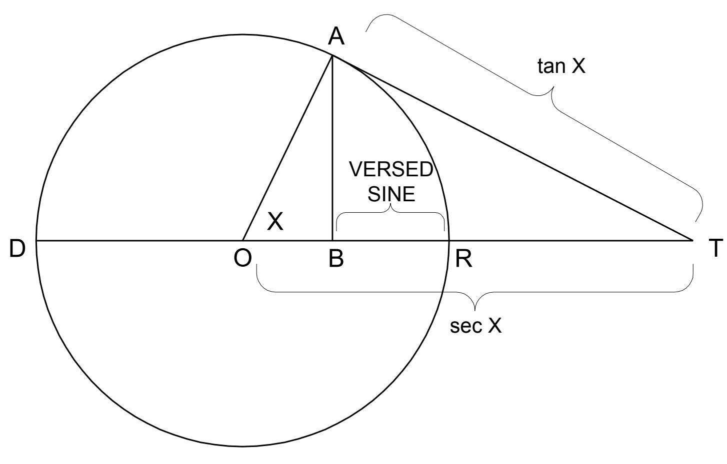
Other lines may be drawn which also grow and shrink together with angle x at the center of our unit circle. If we draw a tangent from A, cutting the extended radius OR at T, then AT is the tangent corresponding to $\angle x$, written:

$$AT = \tan x$$

And OT is the secant (from "secare", "to cut"), i.e. the particular line cutting the unit circle which corresponds to $\angle x$, written:

$$OT = \sec x$$

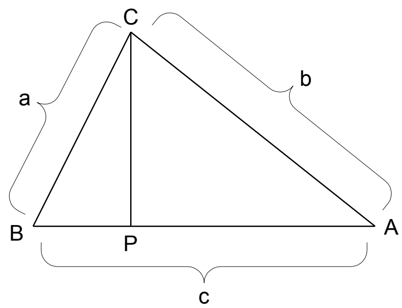
And BR, the segment of the radius leftover from the cosine, is called the "versed sine" of x (i.e. like another type of sine which is "versed," that is, turned 90°). Really no one talks about the "versed sine" these days, since it is nothing other than 1 minus the cosine (OB). Newton calls the "versed sine" the "sagitta," or "arrow," since it looks like an arrow in a bow, although he uses "versed sine" only in reference to the unit circle, and "sagitta" even for other kinds of curves such as conic sections.

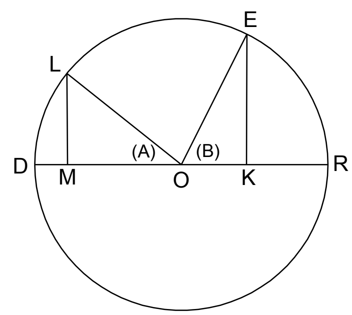


6. THE LAW OF SINES

Kepler uses one rule about sines over and over in order to solve triangles. The rule is called the **Law of Sines**. The Law states: In any triangle having angles A, B, C, and their opposites sides a, b, c, the sines of any two angles are as their opposite sides.

e.g. sin A : sin B = a : b



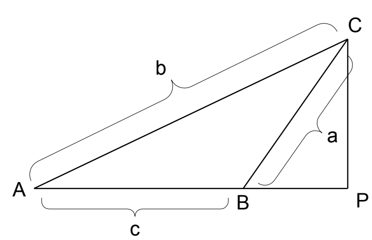


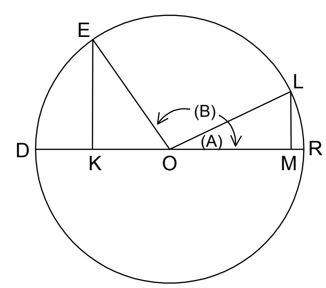
To see why, drop CP perpendicular to AB.
In our unit circle, center O,
draw ∠EOK = ∠B and drop EK perpendicular to diameter DR.
draw ∠LOM = ∠A and drop LM perpendicular to diameter DR.

Now △MLO is similar to △PCA, so	ML : LO = PC : CA
or (since LO = EO)	ML : EO = PC : CA
but △KEO is similar to △PCB, so	<u>KE : EO = PC : CB</u>
Therefore	ML : KE = CB : CA [1]
i.e.	sin A : sin B = a : b

Q.E.D.

QUESTION: What if ∠B were obtuse, so that CP fell *outside* △ABC? Does that affect the conclusion?





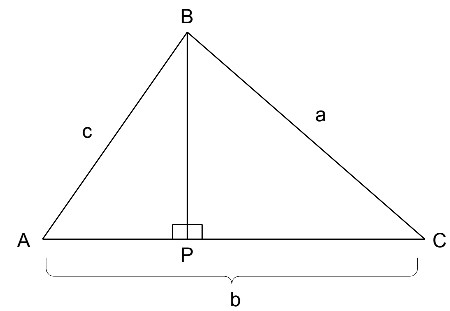
[1] See Euclid 5.23, or 6.16.

7. THE LAW OF COSINES

Another trigonometric rule which can be useful in the study of Kepler is the law of cosines. There is also a law of tangents, but for now we will content ourselves with a quick proof of the law of cosines, thus:

Given: Triangle with angles *A, B, C* and the sides opposite these *a, b, c*

Prove: $a^2 = b^2 + c^2 - 2\,bc \cos A$



Let's assume we have an acute triangle ABC, so that when we drop BP perpendicular to AC, it falls inside the triangle. Also, when we write X^2, let this mean the number we get when we multiply the numerical value of X by itself, rather than some area.

Now $BC^2 = CA^2 + AB^2 - 2\,CA \cdot AP$ [a numerical version of Euc. 2.13]

or $a^2 = b^2 + c^2 - 2\,b \cdot AP$

$a^2 = b^2 + c^2 - 2\,b\,[\,(c/c) \cdot AP\,]$

$a^2 = b^2 + c^2 - 2\,b\,c\,[\,AP / c\,]$

Now imagine △BAP inside our unit circle, with A at the center. Wherever that circle intersects AB (extended, if necessary), from there drop a perpendicular to AP (extended, if necessary), and we will have formed the sine and cosine triangle for angle A. And that triangle will obviously be similar to △BAP. So

$AP : AB = \cos A : 1$

i.e. $AP : c = \cos A : 1$

If we treat these simply as numbers, this means that $AP \div c = \cos A \div 1$.
In other words, $AP \div c = \cos A$, or $AP / c = \cos A$.

Thus $a^2 = b^2 + c^2 - 2\,bc \cos A$

Q.E.D.

8. NOTE ON THE LAW OF COSINES

Notice that we would have a problem if a^2 meant "the square on side *a*," since in that case the conclusion would make no sense. It is all very well to have a square equal to the sum of two other squares minus some *area*, but what does $2\,bc\cos A$ mean? "Cos *A*" is itself a line length, just as *b* and *c* are. So if we take the three together, doesn't that give us a volume? On the other hand, if we only mean the product of their numerical values, we cannot subtract a number from an area, so the previous things, like a^2, must also be taken as numerical values.

And although finite numerical values for things like sines and cosines are rarely exact, there are two ways to deal with that: (1) As we did in Ptolemy, we can say "take them as exact as you need," and the rule becomes only truer as you get more exact values, and we can get it to be true enough for the purposes of astronomy, or we can (2) Learn how to "multiply" not just numerical values, but the straight lines themselves, in all their exactness—which Descartes explains in his *Geometry.*

9. THE LAW OF COSINES FOR OBTUSE TRIANGLES

The proof for the Law of Cosines does not change much if we have an obtuse triangle, with ∠CAB obtuse so that the perpendicular from B to AC falls outside the triangle. Drop BP perpendicular to CA extended.

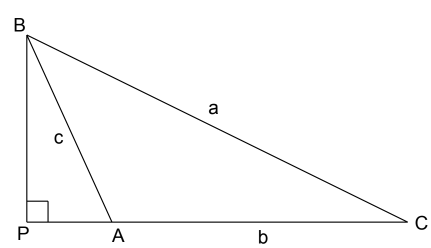
Now $BC^2 = CA^2 + AB^2 + 2\,CA \cdot AP$ [a numerical version of Euc. 2.12]

or $a^2 = b^2 + c^2 + 2\,b \cdot AP$

$a^2 = b^2 + c^2 + 2\,b\,[\,\mathrm{c/c} \cdot AP\,]$

$a^2 = b^2 + c^2 + 2\,b\,c\,[\,AP / c\,]$

As before, *AP / c* is numerically equal to cos *A*, but since angle *A* is obtuse, therefore its cosine is negative (see above, "4. THE DEFINITION OF COSINE."). That means we cannot replace *AP / c* with cos *A* without also changing the sign of the last term in the equation above, thereby falsifying the equation. So, if we wish to make the replacement and maintain the equation, we must also multiply the last term by –1, so we have



$$a^2 = b^2 + c^2 - 2\,b\,c\cos A$$

So the law is the same for acute triangles and obtuse ones.

KEPLER

DAY 36

KEPLER'S INTRODUCTION TO THE *ASTRONOMIA NOVA*

Among all Kepler's writings, perhaps his introduction to *Astronomia Nova* was the most widely read and influential in his own time. It is a rather substantial essay in both content and length (about 30 pages). Let's take a tour through it now, in order to prepare our minds for the difficult matter of the book itself.

THE DIFFICULTY OF BOOKS OF ASTRONOMY

Kepler begins his introduction by noting the difficulty of reading and writing books of astronomy. The mathematics can be very obscure if offered in purely mathematical form and without explanations—but with the necessary explanations, it becomes verbose and tedious. In the book itself, he provides a good deal of explanation (although it is not always the clearest!), so in order to compensate for the undue length, he chose to provide a series of summaries of all the chapters at the outset of the book. Often, if we find ourselves lost in the details of some chapter, wondering what on Earth Kepler is saying, when we turn to the summary we see what he is up to.

There is also some difficulty with Kepler's vocabulary. It is not entirely the same as the vocabulary we learned from Ptolemy, even when he is talking about the same things. And he does not always explain his terminology in the particular places where we are trying to understand what he says. For example, in this introduction, Kepler mentions the "first inequality" and the "second inequality." What do these mean? Their meaning seems to be as follows:

"First Inequality" = "Zodiacal Anomaly"

"Second Inequality" = "Heliacal Anomaly"

The "second inequality" refers to the fact of retrogradation at solar opposition (for an outer planet) and at inferior conjunction (for an inner planet).

And the "first inequality" refers to the fact that the planets move with different speeds in different parts of the zodiac (and their retrogradations are of different magnitude, and, for inner planets, the sums of their elongations from the Sun differ).

THE NEED FOR THIS INTRODUCTION

Kepler then explains that even his summaries will seem prolix to many readers. He calls his work a "labyrinth", and compares his summaries to a "Gordian knot." Part of the reason for this is that he presents the *Astronomia Nova* more or less as a record of his own thought processes and discoveries. He did not go back and remove all the false starts and dead ends, but retained them, with the idea that these might show others not only what he discovered, but how he discovered it. This has the obvious disadvantage of being long and confusing and often frustrating—we can find ourselves struggling for hours or days to understand what Kepler is saying in some chapter, only to find out he abandons it in the next! But there are advantages, too. We get some sense of what a brilliant mind does when it is in the process of making important discoveries. And aside from how illuminating that can be, it is also exciting to watch. There is drama in the *Astronomia Nova* which is altogether absent in the *Almagest*. At times, one feels one is beside Kepler himself, assisting him. One shares his keen desire to unravel the riddles of the motion of Mars.

So, for those who will find even his chapter summaries difficult to follow, he has assembled this introduction, in which he draws together some central ideas which are left a bit scattered in the main work. He also offers the reader the choice of trusting his mathematics, or following him through all the steps. And he warns us that at times he indulges in probable reasoning or conjecture—which is only natural in a work of discovery.

HIS MAJOR PREDECESSORS

Kepler next divides the major schools of astronomical thought:

One kind of school of thought treats each planet separately, and gives each its own causes for appearing to us thus and so. The main representative of this way of thinking is PTOLEMY.

The other type of school, which seeks a single common cause of all the various properties in the planetary motions, is divided into COPERNICUS, who makes this cause the motion of the Earth, and TYCHO BRAHE, who makes it the motion of the Sun.

He notes that these three schools are equivalent to within "a hair's breadth." They produce the same appearances.

THE AIM OF THE BOOK

After this brief description of the models of Ptolemy, Copernicus, and Tycho Brahe, Kepler presents the goals of his book.

The primary goal is to reform astronomy so that the calculated positions of planets at given times fit the observations better. He notes that in August of 1608 Mars was nearly 4° off the position it should have had according to calculations based on the Prutenic tables. (These tables were from an ephemeris produced by the astronomer Erasmus Reinhold which was published in 1551.) To give a sense of just how far off that is, the full moon's diameter is about ½ ° wide. So Mars was appearing almost eight full moon's breadths away from where it was supposed to be according to those tables. That is a gross error.

The secondary goal is to set astronomy on physical foundations, which he says is necessary, anyway, to get results to agree perfectly with the data. So he differs profoundly from Ptolemy and Copernicus by dropping his commitment to the old axiom of circular motion, and committing himself instead to exact correspondence with the observations, and making his system physically intelligible. So now "astronomy" is not so much a branch of mathematics as it is a branch of physics, or it is becoming so.

The two goals, then, are pursued together throughout the book, not in separate parts.

FIRST STEP TO SETTING ASTRONOMY ON PHYSICAL FOUNDATIONS

Kepler next begins summarizing the main steps by which he reaches his goal. These are four:

(1) Establishing that the planes of the planetary orbits intersect in the Sun.
(2) Establishing that there has to be an equant in the solar theory.
(3) Establishing that the eccentricity of Mars's equant is exactly bisected.
(4) Establishing that Mars's orbital path is not a circle, but an ellipse.

The first of these means he will show that the planes of the planetary orbits all intersect *in no other place than the very center of the solar body*. This is a correction of Copernicus and Brahe, who both made the mean Sun (rather than the physical Sun, the solar body) the point of reference. This is part of Kepler's program to banish mathematical fictions having no physical significance from his understanding of the heavenly motions. Mathematical points cannot attract or push or even serve as desirable objects for other things to pursue. They cannot supply a satisfying *physical* account of the motions of heavenly bodies.

He notes that Braheans could say to him "We place the point of intersection of the planetary orbits near the Sun, not in the Sun itself, and we do just fine, thank you very much." So he needs to show that placing it in the Sun itself will give him the same appearances. Similarly for Ptolemy. This is the work of Chs.1-6.

In the next part of the book (Chs.7-21) he says he will take up the orbit of Mars, and shows that his method (of placing the intersection of the planes, etc., in the physical Sun) works not worse, but better, in agreeing with the appearances, than the methods of Ptolemy and Brahe. And while there is much agreement with appearances for all the methods, his alone can be made to agree with physical causes. He will also introduce certain observations which the old methods cannot match, but which his method matches most beautifully. And thereby he demonstrates that Mars's line of apsides passes right through the solar body itself, not just nearby. He will also show that this works not only for longitudes, but also for latitudes.

He will show certain deficiencies in the theories of his major predecessors:

• He will show that Ptolemy's planets cannot move about the geometric centers of their epicycles with uniform motion. Rather, they move uniformly around some other point.

• He will show that Copernicus's orbit of the Earth does not have its geometric center coincident with the point around which the Earth moves uniformly.

• And he will show that Tycho Brahe's circle on which the common point or "knot" (tying together the planets' orbits) moves also does not have its geometric center coinciding with the center of uniformity of motion.

SECOND STEP TO SETTING ASTRONOMY ON PHYSICAL FOUNDATIONS

Next Kepler describes the next main step he took toward setting astronomy on physical foundations: establishing that there has to be an equant in the solar theory. This is something nobody before him thought, not even Tycho Brahe. Ptolemy, for instance, introduced an equant only into planetary theory, and not into the solar theory. But for Kepler, there is something fishy in this. Kepler is a Copernican, which already fits with his insistence upon *physical* reasons for the heavenly motions. The Sun, after all, seems to be the mightiest and most influential body in the heavens. So all the planets should go around it—we should not make it go around any of them, not even us. That means we are going around it (in order to preserve the appearances), as Copernicus said. But then we are just another planet like any other! And if there are equants in all 5 planetary theories, then it makes no sense that there should be none in our own. Kepler went about seeking an equant in the motion of the Sun (that is, in our motion around it), *and he found one*. That is a strong blow to Ptolemy, and a point for Copernicus, that is, for heliocentrism.

In general, it is a goal of Kepler to argue in favor of the Copernican model, since that alone, he sees, can make physical sense of the heavenly motions. Here in the introduction, Kepler summarizes some of the arguments he musters in favor of the heliocentric model. First, Ptolemy is certainly condemned, he says, because Ptolemy would have as many theories of the Sun as there are planets. He does not make any planets go around the Sun, and as a result, each planetary motion, somehow tied to the motion of the Sun, has its own reason for being tied to the solar motion, and practically has its own Sun. But Brahe showed that a single solar theory suffices for all the planets, and we can make them all go around the same point. And in that semi-heliocentric model, we are practically with Copernicus, except that we make the Sun go around the Earth, carrying all the planets with it!

But Copernicus is more compatible with celestial physics than Brahe is. Kepler says this is true for many reasons:

(1) Brahe himself has shown that there are no solid orbs, no crystalline spheres on which the celestial bodies are carried. And yet he has the planets making circles around the *moving* Sun. Why are they doing that, if they are not embedded in nesting spheres concentric with the Sun?

(2) If the Sun is moving, how does each planet keep making a nice nearly-circular orbit around the Sun? If the planet is alive, it seems as though it has to attend to a lot of things to stay in that type of complex motion—running circles around someone else who is also running in a circle is not easy. But if we say the Sun is sitting still, then we have a simple path for the planet, one which can be explained by physical rather than animate powers—in particular, Kepler thinks it is due to magnetism.

(3) Kepler will show that if we allow that the Earth is moving, then it follows that it speeds up as it nears the Sun and slows down as it goes away from the Sun (on its eccentric orbit). That fits nicely with physical causes. But if we make the Sun move instead, we have to say it speeds up near Earth and slows down away from it—Kepler notes that this would follow still, even if we allow that the Sun moves, since this irregularity is the effect of the equant, which he has shown must be introduced into the solar theory, whether we like it or not. Why should that be? Is the Earth acting on the Sun? That seems backwards, since the Earth is so much smaller and less powerful than the Sun. This gives rise to the physical conjecture that the Sun itself is the source of the motion of the five planets.

(4) This idea in turn gives rise to the idea that the Sun itself is sitting still. It is the cause of motion in the planets, so it is most likely (Kepler says) that the Sun does not move.

(5) Otherwise we say that the Earth sits still, and then what will be the physical cause of the motion of the Sun and its enormous burden of the five planets circling it? Is the Earth doing that? Preposterous!

(6) The periods of the planets are yelling out to us that the Earth is in motion around the Sun. Mars has a period of 687 days, Venus has one of 225 days, while the Earth-Sun

circuit (whichever one is moving) has a period of 365 days, which is between those other two periods. Doesn't that suggest that the Earth-Sun circuit lies between the orbits of Mars and Venus around the Sun? And hence that this circuit is a circuit of the earth around the Sun, not the reverse?

(7) The very brightness and size of the Sun suggest that it is the physical seat, somehow, of the motions of the planets, and hence is at rest at the center of the world.

Kepler also considers and overthrows some objections to the motion of the Earth. The first of these is that the Earth is heavy, and therefore rests at the center of the world, a mathematical point, and can't move away from there. Kepler asks how a mathematical point, where there is no body other than the Earth to attract the Earth there, can have the power to do anything at all. It is a non-entity, a mere mathematical thing. He says that the heavy bodies are not attracted to the center of the universe because it is the center of the universe, but because it is the center of a "kindred body." He has the notion that the Earth and Moon are "kindred bodies," which is why the sea is attracted to the moon (hence the tides). If the Earth and moon were not held back from each other by some force, he says, they would descend towards each other. (If only he had taken this a step or two further, and seen that all bodies in the universe are "kindred bodies" in this way, he would have discovered universal gravitation before Newton!)

Another objection to the Earth's motion he considers was also found in Ptolemy, namely that bodies thrown up into the air, or jumping up, should be left behind by the motion of the Earth. But Kepler says that the Earth has a grasp on these objects (even as it attracts the moon), and that grasp would not weaken considerably unless they were removed from the Earth a distance which was perceptible in relation to the Earth's radius. But no familiar projectiles get that far from the Earth.

Still another objection, not found in Ptolemy, is drawn from the Scriptures. Kepler notes that most people who refuse to assent to Copernicus do so because they think the Scriptures say that the Sun goes around the Earth, not the reverse. One familiar passage they have in mind is Joshua 10:12 ff, where the Sun miraculously stands still. Kepler replies that

> Thoughtless persons pay attention only to the verbal contradiction, "the Sun stood still" versus "the Earth stood still," not considering that this contradiction can only arise in an optical and astronomical context, and does not carry over into common usage. (William H. Donahue translation.)

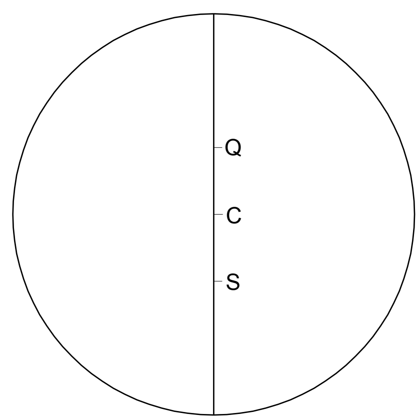
In common speech, "the Sun is rising" does not mean "I'm a geocentrist," but only that the Sun is appearing to rise over the horizon. So too "the Sun stood still in the sky" does not mean "its stopped its movement around the immobile Earth" but only "it stopped its apparent course across our sky." And the Scriptures are not addressed to astronomers, but speak of such phenomena in accord with common usage. Joshua just needed the day to last a little longer, which purpose would be served just as well by stopping the Earth's spinning for a while as it would be by stopping the Sun's orbit for a little while. And either event would be described by the words "the Sun stood still" by any ordinary observer, whether a geocentrist or a heliocentrist. Kepler considers other scriptural passages which use geocentric images, and shows that none of them is opposed to Copernican theory. In general, Kepler does not find any physical or astronomical doctrine in the scriptures.

Following this theological defense of the Copernican model, Kepler gives two kinds of advice to his readers, one kind for astronomers, another kind for idiots. To astronomers, he recommends that they learn to see God's goodness and providence not only in the seeming stability of the Earth, but also in its motion. To idiots, that is, to those who are too stupid to understand astronomy, or else too weak in their faith to agree with Copernicus and still believe in the Scriptures, he advises that they put down the book and mind their own business, namely the business of scratching away at their own patches of dirt in the world. They should not try to understand things above their understanding, but should praise God by looking upon his creation just with their eyes, since it has not been given to them to look upon it with the more penetrating insight of the mind.

THIRD STEP TO SETTING ASTRONOMY ON PHYSICAL FOUNDATIONS:

Establishing that the eccentricity of Mars's equant is exactly bisected.

What does this mean, and why does it have to do with physical considerations of the heavenly motions?



If we draw the actual path of Mars around the Sun, which is at S, and bisect the orbit's line of apsides (i.e. the line through aphelion and perihelion) at C, then ask ourselves where we must place the point around which Mars sweeps out equal angles in equal times (or most nearly does), where will this be? It will be at Q, where QS is exactly double CS. Kepler notes that both Brahe and Copernicus had doubts about this. Copernicus wanted to get rid of all such equants. Brahe seemed to have doubts about where it should be located, and he did not think that the solar theory needed one. Kepler will show them to be wrong about these things.

And what is the physical significance of this? Why is Kepler so keen on equant points? He will show that every planet requires such a point, and that in every case "the eccentricity of the equant is exactly bisected," that is, its distance from the Sun is bisected by the geometric center of the orbit (or the midpoint of the line of apsides). But if the astronomical data forces us to say this about every planet, we are also forced to say that the planet *speeds up near the Sun* and also *slows down away from the Sun* on its own orbit. This becomes plain by drawing a line through Q at right angles to the line of apsides. That line, together with the line of apsides itself, will divide the orbit into two small parts on top and two longer arcs on the bottom, near the Sun. But since the angles thus formed about Q are equal, the planet will spend equal times in those unequal arcs, going through the longer ones near the Sun in the same time as it takes to go through the shorter ones up near Q. Hence the planet goes faster when it is near the Sun, slower when it is further away—*which is a strong indication that the Sun is the cause, somehow, of the motion of the planet*.

FOURTH STEP TO SETTING ASTRONOMY ON PHYSICAL FOUNDATIONS:
Establishing that Mars's orbital path is not a circle, but an ellipse.

This is the main work of the *Astronomia Nova* with which we shall concern ourselves. Kepler spent enormous amounts of time and mental energy trying to determine exactly what shape the path of the planet Mars was. He tried all kinds of ideas that failed. Finally, he established that it is perfectly elliptical. When one sees just how slightly elliptical it is, and how near it is to being a perfect circle, one is moved with admiration for the precision and insight of Kepler. We will see this better later, with diagrams drawn to scale.

He also spent a good deal of time and energy trying to explain planetary motion in terms of magnetic influence from the Sun. His ideas on this score are both obscure and finally incorrect, so we will focus more on his determination of the elliptical shape of Mars' orbit.

Why Mars? One of the reasons for this choice is that Kepler was sure that the actual shape of the planet's path had to be physically significant. Since there were no more "solid orbs," thanks to the work of Tycho Brahe, who proved they cannot possibly exist (the comets seem to move by laws very similar to those of the planets, and yet they pass through the supposed "orbs" of the planets!), we have no special reason to believe that circular motion is particularly important in the heavens. That particular form of the "Astronomer's Axiom" is dead, so far as Kepler is concerned. Very well, then: if there are no spherical mechanisms up there, and so we have lost all physical reason for thinking the orbits are somehow circular or products of circles, what shapes are those orbits? And won't that tell us something about the causes producing those orbits?

About that Kepler is absolutely correct. But while he succeeds in finding out the shapes of the orbits, and while he is right that these orbits indicate that the Sun is somehow responsible for them, we have to wait until Newton before we can discover exactly what sort of influence the Sun has on the planets so as to produce their orbits.

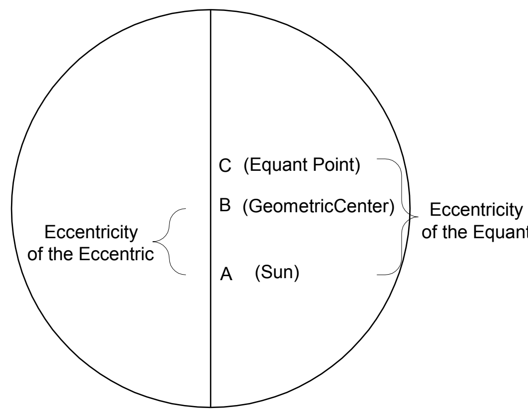
KEPLER

DAY 37

Summary 19
Chapter 19

As with Ptolemy and Copernicus, we will be looking through selections of Kepler's book, and not the entirety of it. Today's main matter will be drawn from Chapter 19 and from Kepler's summary of that chapter. But first, a brief summary of some of the matter from Chapter 16.

In this chapter we are introduced to Kepler's variation of a Ptolemaic hypothesis (which itself is quite simple and lovely), a version which he says he makes "in imitation of the ancients." We came very near it at the end of our study of Copernicus, where we learned how simple a model Copernicus *could* have had, if he had been able to stomach an equant point. Recall that once we say the Earth orbits the Sun, we can (while remaining equivalent to Ptolemy) make the orbit of a planet *simply be a circle* (with no epicycles at all!). Imagine a circle with geometrical center B, the Sun at A, and an equant-point at C. This circle is the orbit of the planet.



GOODBYE EPICYCLES. Why does Kepler begin with this? Partly because of its simplicity, but also, in part, because he has a sense that epicycles simply won't do if we are to take physics seriously. Later, when epicycles come up, he eventually rejects them on physical grounds, although he continues to use an epicycle to the very end as a kind of device representing the physical components involved in planetary motion. Since Brahe has shown there are no orbs, there can be no physical reason why a planet would dance around a mathematical point with no physical reality (it is not the center of a body anymore).

HELLO EQUANT. Why is he, unlike Copernicus, willing to tolerate an equant? Because it makes *physical sense*! If the Sun is the cause of the planet's motion (which is what makes physical sense), and if the Sun is not at the center of the planetary orbit, then the planet must speed up when it gets closer to the Sun, the cause of its motion, and slow down when it gets further away. But then if there is a point around which it sweeps out equal angles in equal times, this cannot be at B, the geometric center of the orbit, but must be at a point higher up. If the planet moved uniformly around B, then it would not speed up or slow down in its orbit at all. To make it go faster near the sun, we must make it go through larger arcs down there and through smaller ones up at the top of the orbit (around aphelion) in equal times. That requires the equalizing point to be somewhere around C.

So Kepler begins with this simple hypothesis, but in pursuing his purpose in the *Astronomia Nova*, he must ground it firmly in the observations. Ptolemy showed that he must place the equant point, C, beyond the center of the deferent (which is now just our orbit), but he assumed that AB = BC without firm proof (Kepler, in an earlier work, accused Ptolemy of random conjecture in this matter, but he here retracts this statement; we will see, in Ch.19, that there is a reason for "bisecting the eccentricity"). After doing this for one planet, he simply assumes it for the rest, and it works out pretty well. Kepler is not satisfied with that. He wants to derive this "bisection of the eccentricity" from the observations. On the basis of about four "acronychal observations" of Mars (i.e. observations of the planet rising while the sun is setting, with the Earth in line between them, i.e. a perfect solar opposition), Kepler begins plugging in guesses at the eccentricity, getting results that don't match the observations, and going back and forth (about 70 times, he says) in this double-iterative process, until he finally gets a ratio of AB : BC which produces the right results. It turns out that the ratio is not that of equality, but rather:

Where the radius of the orbit = 100,000
Then AC = 18572
and AB = 11332
and BC = 7240

This, then, is our starting hypothesis. It is a mere Circle with a Really Unequally Divided Eccentricity. We can refer to it as the C.R.U.D.E. model for short.

SUMMARY 19.

Kepler sums up the matter of Ch.19 thus:

1. While the CRUDE model constructed in Ch.16 agrees well with the motion of Mars in *longitude* near opposition to the sun, Ch.19 shows that this model does not predict the locations of Mars very well in *latitude* near opposition to the sun.

2. He also shows that Brahe's hypothesis has similar problems, and so does Ptolemy's. He does this first in the Copernican form.

3. He does the same in the Brahean form.

4. He shows that the error in latitude results from failure to bisect the eccentricity, i.e. from the equant point and the sun not lying equal distances from the center of the eccentric circle which is supposed to be the orbit of Mars.

5. But if we bisect the eccentricity, like Ptolemy, and therefore get good results in latitude, now all the hypotheses are in error in longitude (near the octants). Kepler says that this is why he was forced to give up on the ancients and search into the matter for himself.

CHAPTER 19

A Refutation, Using Acronychal Latitudes, of this Hypothesis Constructed According to the Opinion of the Authorities and Confirmed by all the Acronychal Positions

6. Kepler's main point: The CRUDE Hypothesis (as I am calling it) is refuted by observations (which are better preserved by bisecting the eccentricity), but also any hypothesis bisecting the eccentricity is refuted by observations.

7. The observations he uses are "**acronychal**," i.e. "night-rising," i.e. when a planet, like Mars, rises just as the sun is setting, and Mars-Earth-Sun are in a nice straight line. What is the advantage or importance of such observations? Well, whether you think Mars orbits the Sun or orbits the Earth, you are looking at its true position in longitude and latitude at these times. So we level the playing field when it comes to the competing hypotheses. We assume little, observe much.

8. Here is the basic story:

Ptolemy bisected the eccentricity, because the acronychal latitudes of the planets required this.

Neither Brahe nor Kepler bisected the eccentricity, because of the kinds of observations discussed in Ch. 16 (i.e. in order to match acronychal longitudes of Mars).

Copernicus, although he rejected the equant point altogether, retained elements in his system which corresponded to a perfectly bisected eccentricity in Ptolemy, and Kepler says the reason for this is that Copernicus did not make much use of observations! But Tycho Brahe "balked at this," i.e. like Kepler he wanted to establish the ratio of eccentricity through observations, but then discovered that planetary motion in longitude required a different ratio.

But when this new ratio of the CRUDE hypothesis was contradicted not only by the acronychal latitudes—which that hypothesis was always bad at matching—but also by observations of the planet in other positions relative to the sun (which are affected by "the second inequality", or the heliacal anomaly of a planet), Tycho turned his attention to the moon instead. Tycho Brahe *gave up* when he saw that bisecting the eccentricity wouldn't work, but neither would the other ratio which the observations seemed to require.

9. "I meanwhile stepped in," says Kepler.

If we assume that the orbit is a simple circle and that there is an equant point, then the four observations require that $AB \neq BC$. Thus Tycho made them in the required ratio, as opposed to Ptolemy (and Copernicus) who made $AB = BC$.

But if $AB \neq BC$, the hypothesis is contradicted by observations of the latitudes of the planet in opposition. Thus Tycho gave up and studied the moon instead. Enter Kepler.

The latitudes near opposition require that AB = BC, and this is what moved Ptolemy to bisect the eccentricity (and in this Copernicus, who "made very little use of observations," blindly followed Ptolemy). But that hypothesis is contradicted by longitudes of the planet in opposition (near the line of apsides and near the octants, as Kepler notes here in Ch.19).

10. SO WHAT'S WRONG?

Kepler recounts the assumptions involved in the CRUDE model, so that he knows what to call into question in his attempt to start over. One assumption was (a) that the orbit of the planet is a perfect circle, and another was (b) that there is some one point inside the orbit, on the line of apsides, around which Mars describes equal angles in equal times. Kepler concludes that at least one of these assumptions must be false, *for the observations used are not false.*

Note the confidence he has in the observations of Tycho Brahe!

So "Equant" and "Circle" cannot both be true, and maybe both have to be dumped.

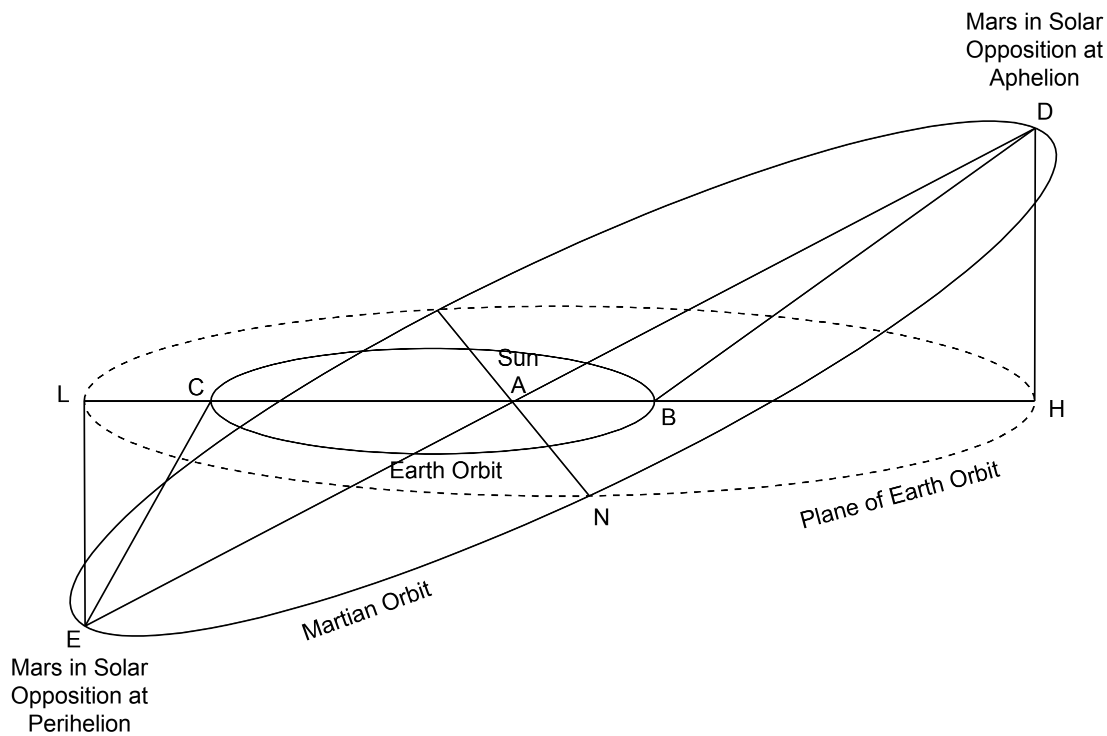
12. DESTRUCTION OF THE C.R.U.D.E. HYPOTHESIS (WITH A NON-BISECTED ECCENTRICITY AS DETERMINED BY LONGITUDINAL OBSERVATIONS OF MARS NEAR OPPOSITION) BY LATITUDES.

Tycho and Kepler's observations of Martian longitudes near opposition, on the assumption of a circular orbit with an equant, geometrically forced them to an eccentricity value of 11,332 in terms of the Martian orbital radius of 100,000 (i.e. the physical Sun and the midpoint of the Martian line of apsides are 11,332 units apart).

We will now see that this hypothesis is destroyed by the LATITUDES of Mars near opposition . . .

Kepler uses line EAD to designate the Martian line of apsides, with A the Sun. Circle CB is Earth's orbit, N is a "**NODE**", a point where the Martian orbit intersects the plane of Earth's orbit. D is Martian aphelion (also the northern limit of its orbit), E is Martian perihelion (also the southern limit of its orbit).

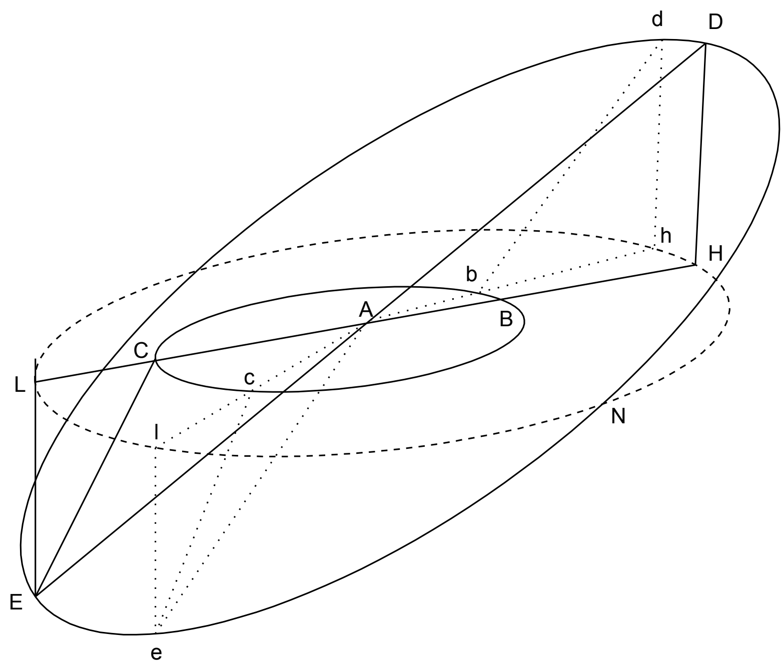
Kepler wants to solve for the eccentricity, which can easily be determined from the ratio of EA : AD. But to do this, he needs observations, and he does not actually have observations of Mars when it is in perfect solar opposition AND exactly at aphelion and perihelion. So he must work from observations of Mars in solar opposition NEAR aphelion and perihelion, and reason back (from the geometry of the hypothesis) to what the values of various angles must be AT aphelion and perihelion. Let "∠DBH" mean the angle right at aphelion, and which we want to calculate. I will use lower-case letters, "∠dbh", to designate the angle we can actually observe.



He says we should conceive a circle of latitude through ADH, i.e. drop DH perpendicular to the plane of Earth's orbit (or the ecliptic). Likewise EL is perpendicular to the ecliptic.

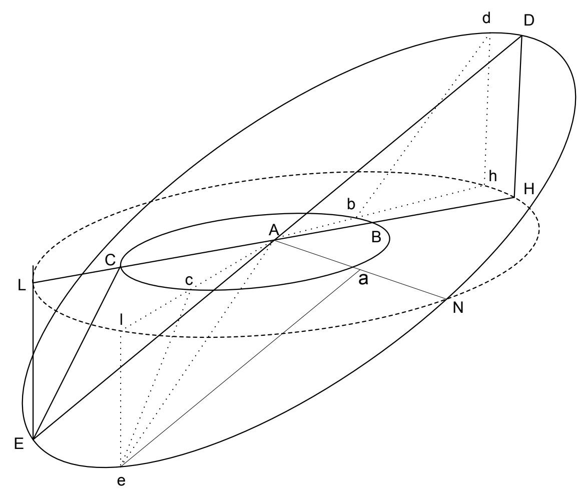
Note: In Ch. 13, Kepler has (by independent means) determined the angle BAD to be 1° 50'. Plainly, in the diagram above, this angle of inclination of the Martian orbit to the plane of Earth's orbit has been exaggerated, to make things easier to see.

But we need to know another angle to use the Law of Sines and begin our calculations—we need angle HBD. But we don't have direct data about that angle. Rather, we have an observation of a Martian solar opposition in 1585 when it is *near* aphelion at D.



So let d be Mars in solar opposition near D (its aphelion), i.e. in 1585, while the earth is at b. This is "four or five degrees from the limit," i.e. from aphelion. So ∠dbh is "apparent," i.e. observed and known. Kepler now says ∠BAD was determined independently in Ch. 13 to be 1° 50', and therefore the value of this angle "four or five degrees from the limit" (where ∠hAH is about 4° or 5°), i.e. the value of ∠bAd, will be somewhat less. He specifies that ∠bAd must be about 1° 49.5', without explaining himself: "*Therefore, at four or five degrees from the limit it will be 1° 49½'*". When he makes a similar move at the perihelion-end of Mars's orbit, however, he is more explicit, so we will deal with it there. For now, let's just assume he has a technique for finding ∠bAd since he is given ∠BAD and ∠HAh.

So now we have both ∠dbh and ∠bAd. But we also know the supplement of ∠dbh, i.e. ∠dbA. Hence we have two angles in △dbA. But that gives us the third angle, bdA, since the sum must be 180°. Moreover, we know the length AB, since that is the Earth-Sun distance (and we always know our position around the Sun, and we have a similar hypothesis for Earth's motion as we do for Mars's).



So we apply the Law of Sines:

sin (∠bdA) : bA = sin (∠dbA) : dA

We solve for the unknown dA, and its length is now given, or at any rate the ratio of bA : dA is given. He says that

if bA = 100,000
then dA = 167,200

He repeats the procedure near perihelion, where Ae is the line near AE:

if cA = 100,000
then eA = 137,380

Above, recall that he had some way of calculating the value of ∠bAd when given the values of ∠BAD and ∠hAH. He does the same thing now to get ∠cAe, except this time he explains himself. To see his technique, draw ea parallel to EA, and intersecting AN at point *a*. Kepler is giving us a proportion of four sines:

sine ∠EAN : sine ∠CAE = sine ∠eAN : sine ∠cAe

He knows ∠EAN, since that is just 90°.
He knows ∠CAE, since that is just the inclination of the Martian orbit.
He knows ∠eAN, namely 64° of difference between the longitude of Mars and its node.

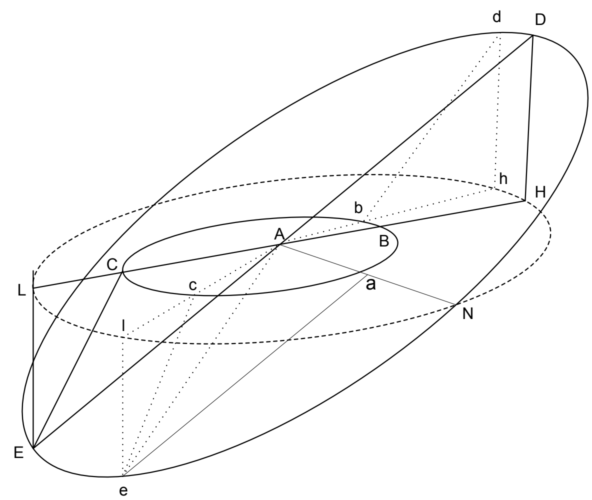
So the only thing he does not know in the proportion is sine ∠cAe, which he solves for, getting 1° 39'.

But how does he know the proportion is true?

Well, since *ea* is parallel to EA, and *el* to EL, thus plane *lea* is parallel to plane LEA, and hence these planes intersect the plane of Earth's orbit in parallel lines, i.e. *la* is parallel to LA. Hence the triangles *lea* and LEA are contained by parallel sides, and are therefore similar.

Hence el : EL = ea : EA

i.e. $$\frac{el}{EL} = \frac{ea}{EA}$$



so $$\frac{el}{EL} = \frac{ea}{EA} \cdot \frac{eA}{eA}$$

or $$\frac{el}{EL} = \frac{eA}{EA} \cdot \frac{ea}{eA}$$

thus $$\frac{el}{eA} = \frac{EL}{EA} \cdot \frac{ea}{eA}$$

so $$1 : \frac{EL}{EA} = \frac{ea}{eA} : \frac{el}{eA}$$

or sin(EAN) : sin(LAE) = sin(eAa) : sin(lAe)

but ∠LAE = ∠CAE
and ∠eAa = ∠eAN
and ∠lAe = ∠cAe

so sin(EAN) : sin(CAE) = sin(eAN) : sin(cAe)

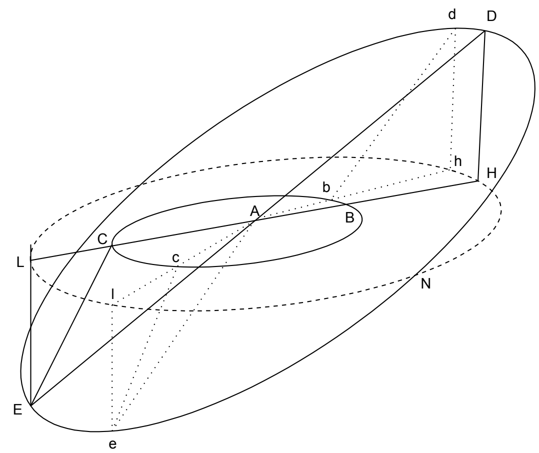
which is the proportion Kepler uses.

So, knowing ∠cAe, we can now determine all the sides and angles of △ecA, just as we did with △dbA above after determining ∠bAd. This means we know the ratio Ad : Ab, and again we know the ratio Ae : Ac. But we also know the ratio Ab : Ac (from solar theory). Hence we know the ratio Ad : Ae, which at least gets us two lines in the plane of Mars's orbit, although they do not lie along the line of apsides. By making further trigonometric calculations, Kepler is able to determine what the corresponding distances must be EXACTLY at aphelion and perihelion. He says that where bA is 97,500 and cA is 101,400, the values of DA and EA in these units will be:

DA = 163,150
AE = 139,000

Now he adds them together, and gets the whole line of apsides:

DE = 302,150



Now cut it in half at K, and
DK = 151, 075
and subtract DK from DA (163,150) in order to get the eccentricity, i.e.
AK = 12,075

He now makes a proportional adjustment, so that the value of DK will be 100,000. This is because DK is the radius of the Martian orbit, which is perfectly circular on the present hypothesis.

If DK = 100,000 (instead of 151,075)
there AK = 8,000 (instead of 12,075)

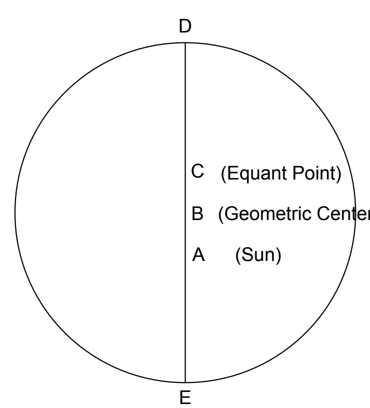
So, if we insist on our C.R.U.D.E. hypothesis, and apply these observations in latitude, those are the values we get. But then again the same hypothesis, together with certain observations in longitude (Ch.16), required us to say

If DK = 100,000
then AK = 11,332

as Kepler says here in Chapter 19.

And therefore the hypothesis has something false in it.

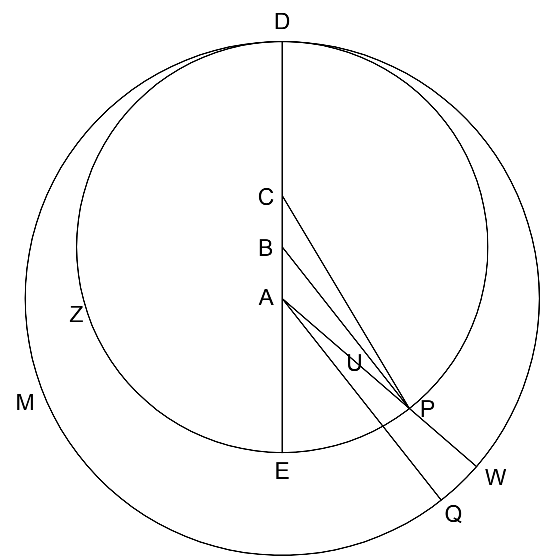
12. DESTRUCTION OF THE "PTOLEMAIC" VERSION OF THE CIRCULAR HYPOTHESIS, i.e. ONE WITH A BISECTED ECCENTRICITY.



We could call this hypothesis C.W.A.B.E., for Circle With A Bisected Eccentricity. Let the circle of diameter DE, center C be the Martian orbit, and let A be the Sun, C the equant, and let the eccentricity be bisected, i.e. let AB = BC = 9282 (where radius BD = 100,000).

Now Kepler brings up a time of solar opposition in 1593, where the Earth is at U, the sun is at A (of course), and Mars is at P. Hence we could observe the direction of the line AP at that time, and it was 12° 16' into Pisces. So if we draw a circle around A as center, with radius AD, then Mars has moved through arc DMW in longitude around the Sun between its last aphelion and its location observed at solar opposition in 1593.

But since the time is given, and the mean speed of Mars in longitude is given, and the orientation of Mars's line of apsides is given, therefore the hypothesis determines what Mars's longitude should be. Let's see if it matches the observation.



What we are given by the theory is:

∠ECP = 11° 3' 16'' (This is given basically by the TIME of the solar opposition, since C is the equant, the point of uniform motion)

and AC = 18564 (the whole of the bisected eccentricity)

and BC = ½ AC = 9282

where BP = 100,000

But this allows us to apply the Law of Sines to solve for ∠CPB thus:

sin (∠BCP) : BP = sin (∠CPB) : BC [Law of Sines, △BCP]

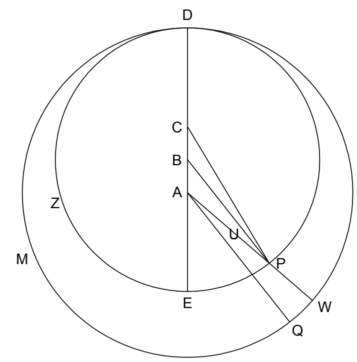
but ∠BCP = ∠ECP = 11° 3' 16''

so the only unknown is sin (∠CPB). So we solve for that sine, and then, using a table of sines, we have the value of ∠CPB, namely

∠CPB = 1° 1' 12''

Kepler calls this angle "**THE PHYSICAL PART OF THE EQUATION**," presumably because the triangle CPB includes angle BCP, which is about C, the point of the planet's uniform motion, and hence a point connected to its true physical behavior. The equant has physical significance for Kepler, since if the planet moves uniformly around C, it does not move uniformly around the Sun, but speeds up in proximity to it, as geometry dictates. And that makes physical sense to him.

But now, knowing both ∠BCP and ∠CPB, we also know their sum, and hence we also know the exterior angle EBP.



∠EBP = 12° 4' 28''

Hence we know also the supplement of this angle,

i.e. ∠DBP = 180° – ∠EBP = 167° 55' 32''

and so now we also know the arc DZP as measured around point B, since it is 360° minus the angle DBP,

i.e. arc DZP = 192° 4' 28''

which Kepler designates as 6^S 12° 4' 28'', i.e. "six signs", or 180°, plus 12°, plus 4' 28'' (each sign of the zodiac is 30°).

Now let's look at △BAP:

$AP^2 = AB^2 + BP^2 - 2AB \cdot BP\cos ABP$ [LAW OF COSINES][1]

But AB = 9282 (the theory we are testing says so)
and BP = 100,000
and cosABP = cosEBP = cos 12° 4' 28''

So the value of AP is now given (by a table of cosines).

Now, using the Law of Sines in △BAP, we have

sinBPA : AB = sinABP : AP

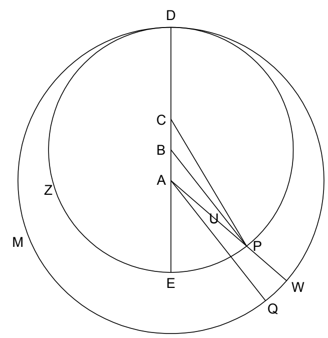
and the only unknown is sinBPA. So we solve for that, and using a table of sines, we will have the value for ∠BPA,

i.e. ∠BPA = 1° 13' 26''

Kepler calls this angle "**THE OPTICAL PART OF THE EQUATION**," presumably because it includes the line AP, which is our line of sight (since we are at U).

[1] Kepler actually uses the Law of Tangents.

Now draw AQ (out to the circle of radius AD) parallel to BP.



Now ∠EAQ = ∠EBP [since AQ is parallel to BP]
but arc DZP = 180° + ∠EBP
and arc DMQ = 180° + ∠EAQ
thus arc DZP = arc DMQ [i.e. in degrees, not linearly]

But Mars's longitude around the Sun, measured from aphelion D, is arc DMQ *plus* angle QAP,

i.e. longitude = arc DZP + ∠QAP

But ∠QAP = ∠BPA [again, by the parallels AQ, BP]
and ∠BPA = 1° 13' 26'' [as determined by theory, above]
so longitude = arc DZP + 1° 13' 26''
i.e. longitude = 6^S 12° 4' 28'' + 1° 13' 26''

But that is the longitude of Mars from aphelion. To get its longitude from the spring equinox (in order to compare it with the observed value), we have to add in the longitude of the aphelion itself (from the spring equinox), namely 4^S 28° 55' 43''

So the total *theoretical* longitude of Mars at the time of solar opposition in 1593 should be:

4^S 28° 55' 43'' [longitude of aphelion]
\+ 6^S 12° 4' 28'' [arc DMQ from aphelion]
\+ 0^S 1° 13' 26'' [arc QW]
11^S 12° 13' 37'' from the spring equinox
or 12° 13' 37'' into Pisces.

But the actual observation was 12° 16' into Pisces. That is an error by a full 3' of arc, which is unacceptably large—especially since we are talking about Tycho Brahe.

13. A GROSSER ERROR.

Kepler adds "This appears more clearly at 17° Cancer in 1582," where the error produced by the hypothesis of bisecting the eccentricity is almost 9' off from the observation.

Here Kepler notes that this discrepancy of 8 minutes, being so small, makes it clear why Ptolemy was satisfied with his bisected eccentricity, and why many people would be satisfied with the C.W.A.B.E. hypothesis. He notes that Ptolemy admitted that he did not get more precise than 10 minutes or a sixth of a degree, in his observations. To put that more concretely, he observed the heavens only to a precision within one third of the apparent diameter of a full moon, no more. But, says Kepler, divine providence gave us Tycho Brahe, who made far more precise observations of the heavens. Being off by 8 or 9 minutes from Tycho's observations means the hypothesis is simply wrong.

Back to the drawing board.

KEPLER

DAY 38

Summary 21
Chapter 21

SUMMARY OF CHAPTER 21

We now move ahead to Chapter 21 of the *Astronomia Nova*. The title of the chapter is "Why, and To What Extent, May A False Hypothesis Yield the Truth?"

Kepler summarizes this chapter thus:

> Causes are sought from geometry that would result in the truth's proceeding from a false hypothesis; and it is shown to what extent this can happen. And this is the end of the second part, in which I have imitated the ancients. (William H. Donahue translation)

CHAPTER 21

What is Kepler doing in this chapter?

1. This chapter investigates what its title promises, "Why, and to what extent, may a false hypothesis yield the truth?" It is very philosophical in nature. Kepler is anxious to defend the basics of the Copernican model, that is, heliocentrism, from the attacks made on it by certain logicians. He makes reference to a certain "axiom of the logicians" which he particularly abhors, namely "that the true follows from the false." These logicians would have it that Copernicus's theory does no more than produce conclusions which agree with the observations, but it is possible for the true to follow from the false—hence the fact that true things follow necessarily from Copernicus's model in no way testifies to the truth of that model.

The axiom he refers to can be illustrated with a silly example:

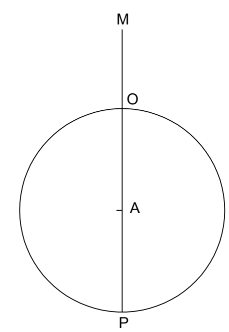
Every Cat is a Giraffe
<u>Every Giraffe is an Animal</u>
so Every Cat is an Animal

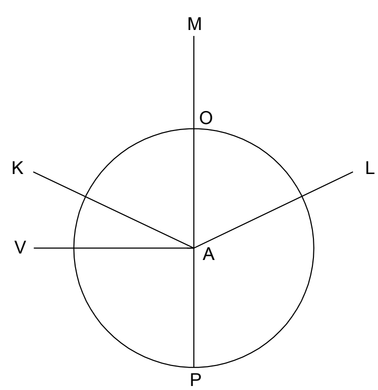
So the true conclusion follows necessarily from the false premises—and therefore the truth of the conclusions of a theory do not prove that the theory is true.

2. Kepler hates that argument. He thinks Copernicus is right in a general way, and that we can know this by arguing from the manner in which it necessarily produces certain consequences which check out. He cannot believe that it might be a mere coincidence that so many of the facts follow necessarily from the heliocentric view, while these same facts are mere coincidences and not at all necessary in Ptolemy's view. Rather, the cause of this is that Copernicus has the truth, and Ptolemy does not.

3. Kepler does not think that what follows from Copernicus's theory is always the truth. We saw, in Ch.19 for instance, that some incorrect things follow from the *details* of his hypotheses.

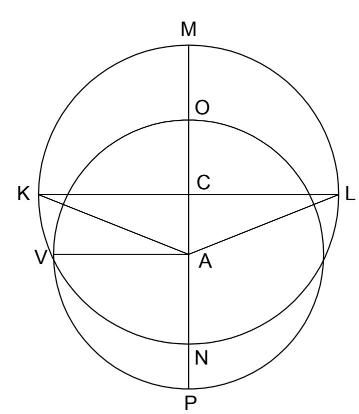
4. He says that false hypotheses may SIMULATE the truth, as far as longitudes are concerned, at least, to within the limits of observational precision. He demonstrates this statement with a rather horrendous diagram, which I will produce in stages, so that it is less confusing.





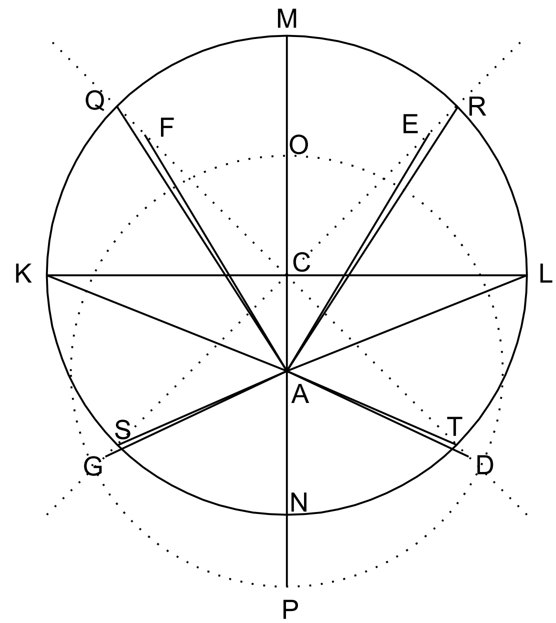
5. FIRST CIRCLE, OP, CENTER A. If a body moves around the observer at A, and it spends equal times on both sides of PM, it could be moving uniformly about center A on the circle with center A, radius AO (but also on an infinity of other circles and curves!).

6. SECOND CIRCLE, MN, CENTER C. But suppose the body spends unequal times in the quadrants of circle OP, but equal times in the angles KAO, OAL, LAP, PAK. Thus the maximum error of our first hypothesis is ∠VAK. If so, we can account for that with a new circle of center C, radius CK, diameter MN, and make the planet move uniformly about center



C now. (He notes that Mars in fact spends unequal times in the quadrants, and the ∠VAK is 10½°.)

So this new circle will correctly represent the location of the planet along the lines AM, AN, AK, AL.

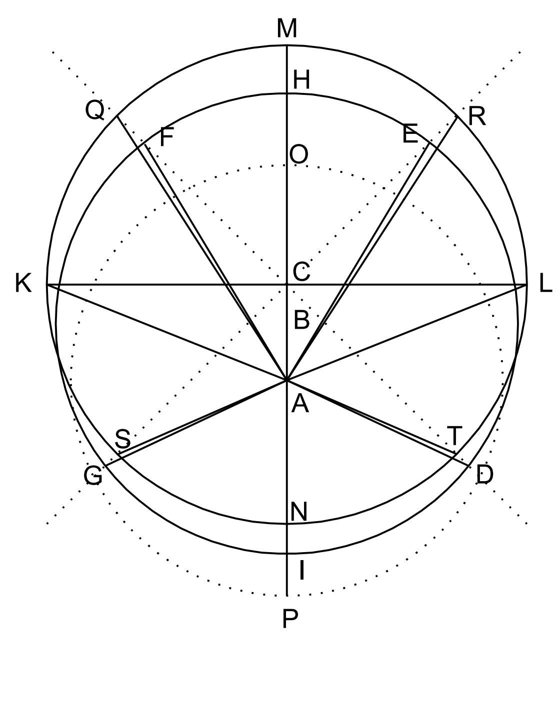


7. THIRD CIRCLE, HI, CENTER B. Now suppose (as is true for Mars) that in the eighths of periodic time, that is, half way through the times when the star is appearing along the lines AM, AN, AK, AL, it does not appear along the lines AQ, AR, AS, AT, that is, along the lines pointing to the eighths of our new eccentric circle, circle MN, but instead is above AQ and AR at AF, AE, and below AS and AT at AG, AD.

Assume, in other words, that the planet does not spend equal times even in the octants of the new eccentric circle. So now he draws yet another circle, now with a center at B, but keeping C, the center of our failed second circle, as the point of uniform angular motion—in other words, it is an equant.

The new facts alone do not force us to this, but rather three other things freely assumed and not forced by demonstration might lead us to introduce this equant-hypothesis, namely (a) that we don't want to disturb our prior hypotheses too much, (b) that we want to keep the point of uniform motion at C, thus fixing the distance AC (on account of the angle KAV in the earlier diagrams), which is not absolutely necessary, and (c) that we insist on keeping the planet's path circular.

So we make a new eccentric path as circle HI, geometric center B, while keeping the point of uniform motion at C. He says that (presumably for Mars) the errors at the eighths of the period *were* 9' in one place, 28' in another, vs. the horrendous ∠VAK (which is 10½°) of our first hypothesis, circle OP. But our new equant hypothesis, circle HI, with C the center of uniform motion, entirely absorbs these little errors, and gets the longitudes exactly right for 8 places of Mars.



8. WHAT ABOUT THE SIXTEENTHS? Now suppose we have an error at the sixteenths! This, he says, will be much smaller than our prior errors. He uses an estimating proportion:

(a) If the error at the 4ths was 10½°, then corrected by a new circle,

(b) but this new circle correct at the 4ths was itself in error at the 8ths by 9' and 28' (in different places),

(c) then, supposing the circle which is correct at the 8ths will be in error at the 16ths by a proportionally smaller amount, we are talking about a new error, in the 16ths, which has the same ratio to 9' that 9' has to 10½°. But 10½° = 630 minutes, i.e. 70 times 9'. So the error was reduced to 1 seventieth what it was before, going from one circle to the next. That means, going to the 16ths, we can expect the error to be somewhere around one 70^{th} of 9', i.e. about 7 arc seconds!

Kepler notes that even just at the sixteenths of the period, we have already entered into hypotheses that will (probably) be wrong everywhere except at 16 places, and yet undetectably so, as far as naked eye astronomy is concerned.

9. His main point seems to be this: so long as we stick to perfect circles and uniform motion, we could easily be making our hypothesis less and less distinguishable in its effects from the appearances, and yet have it still be false throughout. Our hypotheses will be off by too small an error for us to discern by the senses, but never actually be on the true path.

10. Kepler now concludes that it is clear to what extent and in what way the truth can follow from false principles. He appears to be saying that only the true hypothesis produces all of the true points on the orbit, and thus matches with all conceivable observations. False ones, which are retrofitted to some finite number of observations, will generally be right about those (of course), but wrong about all others, although perhaps undetectably so. How, then, are we to discern the true from the false? By coming up with *physical reasons which dictate the geometry of things*. If, instead of merely fitting our hypothesis to observations, we *deduce* what all the observations should necessarily be from physical causes, and then we are not merely right about some of the observations, but all at once we are right about all of them, then we have found the truth. That is the ideal. And Copernicus is an instance of that, as far as his general model is concerned. He said that the Sun is the mighty body in the universe, not the Earth, and so things should be moving around it, not it about the earth. From the physical reason comes forth a single hypothesis about the motions of the heavens, namely that everything moves around the sun—and from that come forth, with necessity, all the basic facts about the anomalies, as we saw back in Copernicus.

KEPLER

DAY 39

Summaries 22, 24
Chapters 22, 24

SUMMARY Ch. 22

1. In this summary, Kepler notes that he will "Begin the whole inquiry anew" in Ch. 22, which makes it very explicit that this book is a record of a discovery process, not an orderly presentation of demonstrative information.

2. He says that he began to suspect an equant must be present in the theory of the Sun, though no one had ever posited that before. He will present a way for finding observations from which to demonstrate the presence of such an equant. What is the significance of the Earth's orbit (or the Sun's, for Ptolemy) having an equant? He said in his introduction that the solar theory (or Earth's motion) is built, whole and entire, into the theory for every planet. For the inner planets, their own orbits are like epicycles, and Earth's orbit is their *deferent* (cf. Brahe's vision, where the [mean] Sun is at the center of the inner planets' epicycles, which are carried along with the Sun, its eccentric orbit around us being their deferents). For the outer planets, their epicycles are Earth's orbit (recall the equivalence proof at the end of Copernicus, showing what he "could have had"). This means that the epicycles of the outer planets have an equant in them, and so does the deferent of the inner planets. Equants galore!

3. The interesting thing about equants (for Kepler) is that they make everything speed up as they approach the Sun, and slow down as they recede from it. They make physical sense to him. (To us, they tend to look like a leftover from ancient astronomy. Really there is no such thing as an equant-point for any planet's orbit, nor is there any reason to believe in any such thing, unless you accept the old axioms.)

So we can conceive of this as a way of answering a possible objection, i.e. "If your *physical* causes make equants necessary, then why would Earth's orbit be an exception?" Answer: *it isn't.*

4. Brahe used to account for the data by making the Earth's orbit (or rather the Sun's around us) grow and shrink, as though it was breathing. That way of thinking about it seems crazy to Kepler. Nature doesn't operate that way. Throbbing epicycles are not allowed.

5. He will begin afresh with the heliacal anomaly. (Recall that the "first inequality" is the zodiacal anomaly, the "second inequality" is the heliacal.) So it appears he will be getting to the equant in the Earth's orbit through the theory of the planets.

SUMMARY Ch. 24

In this summary, Kepler promises to prove the same thing again as was proved in Ch. 22, namely, that there must be an equant in the Sun's (or Earth's) motion. He will demonstrate this from certain observations of Mars, and we will pay more attention to his argument in this chapter than to the one in Ch. 22.

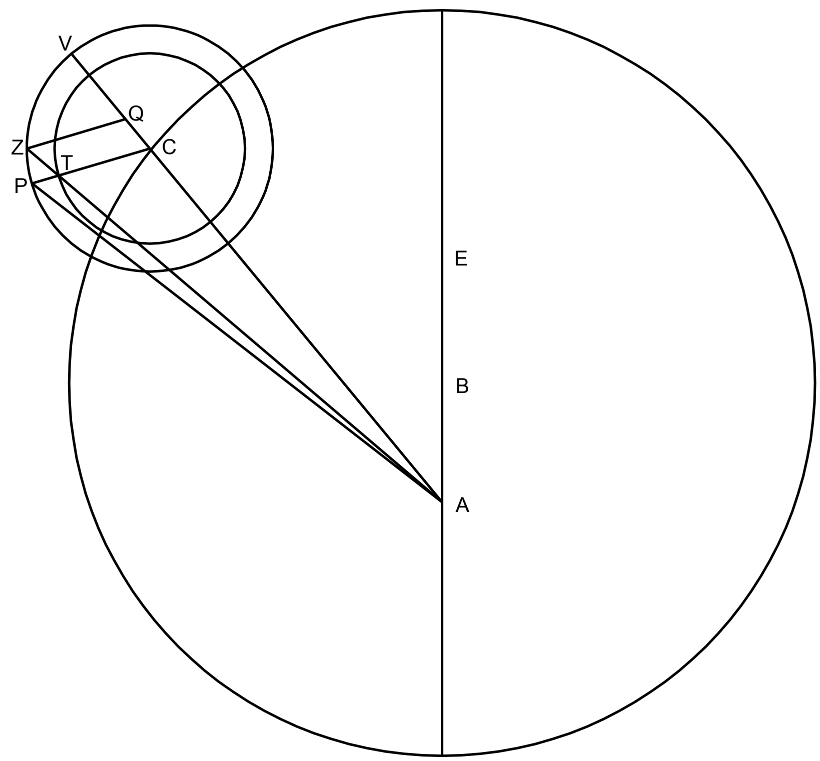
CHAPTER 22

Kepler begins this chapter by noting that in his *Mysterium Cosmographicum*, when he was explaining the physical cause of the Ptolemaic equant (or, equivalently, of the Copernican-Tychonic second epicycle), he raised an objection against himself: if the cause he were proposing were the true cause, then shouldn't it be true universally for all planets? But up till Brahe's and Kepler's time, no one thought the Earth's orbit (or the Sun's) required an equant.

The "physical cause" according to Kepler lies in the Sun, and IF the Sun is the chief physical cause of the motions of the planets, THEN they should move faster when closer to it, slower when further away, and therefore they cannot sweep out equal angles in equal times around it, but must do so (if they do so at all) around some OTHER point. And why can't that point be their own geometrical center, if their center is away from the Sun? Because the Earth is speeding up and slowing down on its own circular orbit, and so does not sweep out equal angles in equal times around its own geometric center. So his physical theory requires something like an equant (if there is to be any point around which equal angles are swept out in equal times).

But Brahe told Kepler that the Earth's orbit is growing and shrinking. That was Kepler's first clue that Earth, too, had an equant after all: Kepler knew that there could be no physical cause of an orbit growing and shrinking. Something else was going on. Let's illustrate the sort of thing Kepler is talking about with a diagram.

Let A be the Sun, B the center of an outer planet's deferent, E its equant, C the center of the epicycle at a given time, AV the line through C, ∠VCP the angle of movement on the epicycle dictated by the time. Hence the star ought to be at P. But suppose it appears along the line ATZ. This could be either because ∠VCP was taken around the wrong point, and should be around Q, a kind of equant in the epicycle (which is what Kepler will say), or it could be because the star is at T, and the epicycle has shrunk from what we knew it to be at other points in the orbit.



CHAPTER 24

A MORE EVIDENT PROOF THAT THE EPICYCLE OR ANNUAL ORB IS ECCENTRIC WITH RESPECT TO THE POINT OF UNIFORMITY

1. This chapter does not prove conclusively that the Earth's orbit "has an equant," i.e. it does not prove that there is a point of uniform angular motion, but only that if there is one, it is not at the geometric center of the Earth's (presumably) perfectly circular orbit.

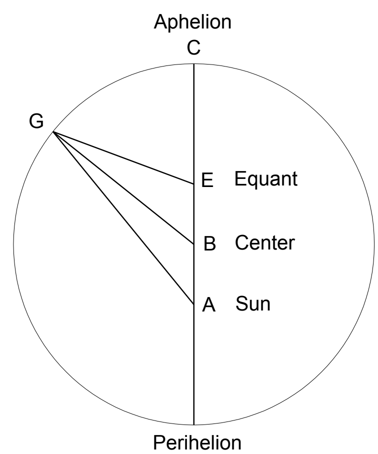
2. In the first sentence of the chapter, he mentions the "anomaly of commutation." Seemingly, Kepler uses "anomaly" to mean any angular motion (he uses "inequality" to designate what we call an "anomaly"). So an "anomaly of commutation" seems to mean just some "angular measure of movement."

The "true" or "equated" "anomaly of commutation" means angular motion around the center of the solar system (the Sun, for Kepler, but Earth for Ptolemy), while the "mean anomaly of commutation" means angular motion around an equant (a point of reference for uniform angular velocity).

3. More vocabulary:

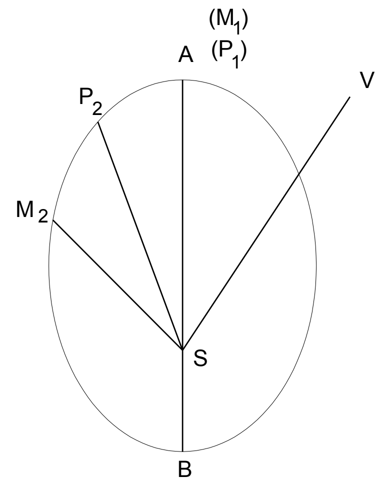
∠CEG = "mean anomaly"
∠CBG = "eccentric anomaly"
∠CAG = "equated anomaly"

∠EGA = "the equation"
∠EGB = "the physical part of the equation"
∠BGA = "the optical part of the equation"



4. Still more vocabulary: "MEAN LONGITUDE."

P is the actual Planet, which moves from P_1 (at Aphelion, A) to P_2.
M is the Mean planet, which moves from M_1 (at Aphelion, A) to M_2.
S is the Sun.
V is the Vernal equinox.



"Mean longitude" refers to the longitude of the mean planet from the Vernal equinox as measured around the Sun. At aphelion, the longitude of the actual planet and that of the mean planet coincide, i.e. $\angle VSM_1 = \angle VSP_1 = \angle VSA$. But elsewhere, they differ, e.g. later on mean longitude is $\angle VSM_2$, which is not the same as the actual longitude of the planet, $\angle VSP_2$.

Mean longitude is therefore like a measure of time in angles, starting from the spring equinox by convention.

Kepler and Brahe were so exact about longitude, that they also took into account the precession of the equinoxes in recording any longitude. This is because "the spring equinox point," the conventional point of reference for longitudes, precesses about 51'' per year—that is Brahe's value, which Kepler uses.

5. BACK TO THE ARGUMENT.

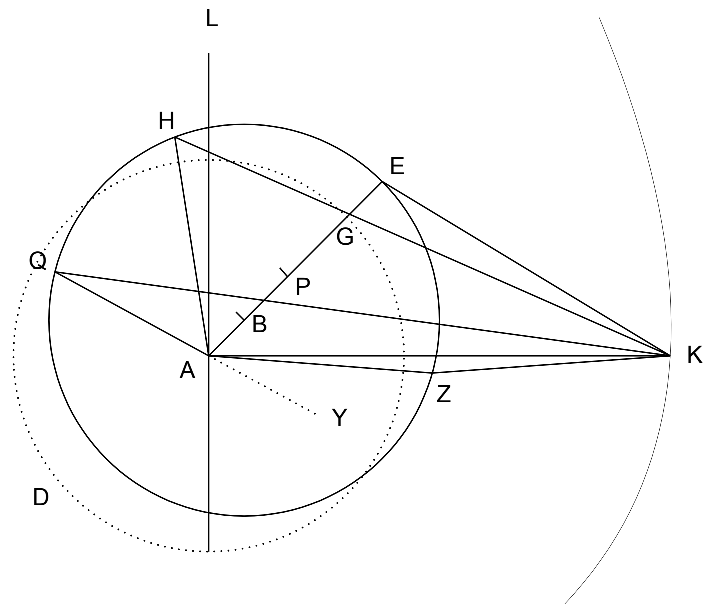
Now we want to see how Kepler establishes that the point of uniform angular velocity, for the Earth, is not the geometric center of its perfectly circular orbit.

By Kepler's physical principles, the Sun is the chief motive force moving the Earth. But Earth's orbit is eccentric to the Sun, and therefore gets closer to it and further from it, and therefore should speed up and slow down on its own circular orbit, and therefore should not travel uniformly about the center of its orbit. Therefore the center of its orbit is distinct from the center of its uniform motion. Therefore Earth has an equant.

But do observations warrant this? Yes . . .

6. SETTING UP THE ARGUMENT.

He works from four locations of Earth on its own orbit, at four different dates when Mars was at the same longitude, very near its node (so that we have no latitudes to worry about, and we can regard everything as being in one plane).



A = Mean Sun = Earth's point of uniform motion.
DG = Earth's presumed orbit.
P = Physical Sun.
Q, H, E, Z = 4 positions of Earth when Mars is at K, near one of its nodes (at 4 different dates:

Q =	Mar 1590
H =	Jan 21 1592
E =	Dec 8 1593
Z =	Oct 26 1595

AL = Mars's line of apsides (through the mean Sun vs. through the physical Sun)

Since Mars returns to K in equal times (K is a real point on its orbit, not just an apparent position from Earth);

- therefore equal times have elapsed between the 4 positions of Earth;
- therefore Earth has swept out equal angles about point A during those intervals (even if in each interval it went more than 360°);
- therefore ∠QAH = ∠HAE = ∠EAZ.

7. THE ACTUAL ARGUMENT.

- We know QA in position from the theory of the mean Sun.
- We know AK in position from the Mars theory correlated to the mean Sun.
- So we know ∠KAY.
- So we know ∠QAK = 180° – ∠KAY.
- And we know ∠AQK, since Q is us, K is the observed position of Mars, A is the calculated position of the mean Sun.
- So we know ∠QKA = 180° – ∠QAK – ∠AQK [Euclid, *Elements*, 1.32]

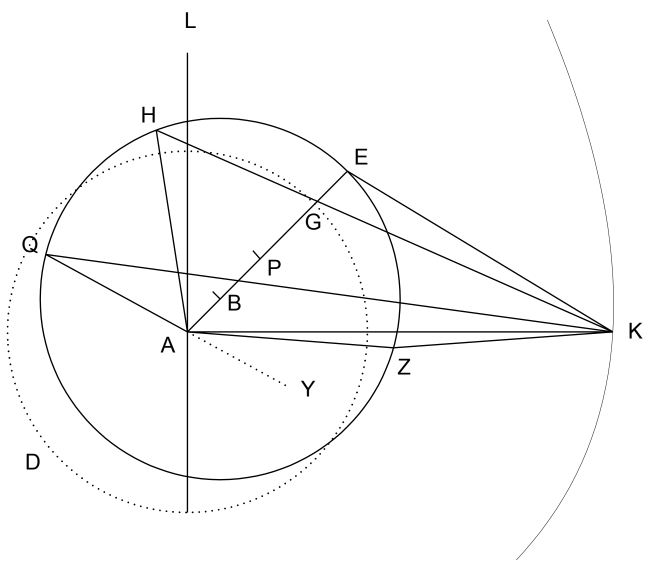
Calling AK = 100,000 [our standard for all 4 measurements], we now use the Law of Sines:

sin AQK : AK = sin QKA : QA

Now solve for AQ in terms of AK.
Repeat for AH, AE, AZ.
Kepler finds *they are unequal.*

- Therefore A, the point around which Earth moves with uniform angular velocity, cannot be the geometric center of Earth's circular orbit (which is more toward B).
- Therefore A is an equant.

Q.E.D.



KEPLER

DAY 40

Chapters 32, 33

CHAPTER 32

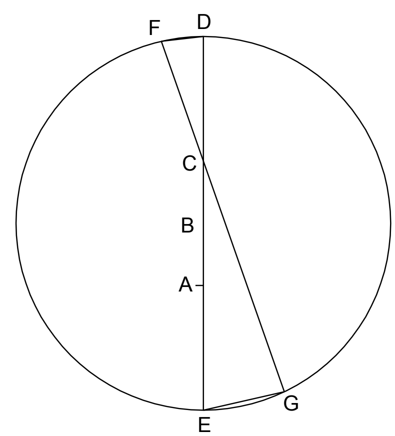
THE POWER THAT MOVES THE PLANET IN A CIRCLE DIMINISHES WITH REMOVAL FROM ITS SOURCE

Up to this chapter, Kepler has shown:

1) The old theories won't work (they disagree with the observations).

2) All the planets, including Earth, have an equant (or at any rate do not move uniformly around the geometric centers of their own orbits), and so he has removed the obstacle to his theory regarding the physical cause of the equant point .

So now he feels justified in advancing his theory of the physical cause of equant-points, namely that the planet speeds up near the Sun, and slows down when further away from it.



One way to argue for this notion is as follows:

Let DFEG be the planet's eccentric orbit,
with center B
aphelion D
perihelion E
the sun A
the equant point C

And let AB = BC, using the "Ptolemaic form" or "bisected eccentricity".

(We have already shown that this model is too simple and does not agree entirely with the facts. But for the moment we are abstracting from its deficiencies and using it to see why Kepler thought that the speeds of the planet were inversely as its distances from the sun.)

Through C draw FCG, intersecting the circular orbit at F very near aphelion, and at G very near perihelion.

Prove: speed of planet in arc DF : speed in arc EG = EA : DA

Now	∠FCD = ∠GCE	[vertical]
so	time of arc DF = time of arc EG	[equal angles about C, equant]

But when two uniform motions are allowed to go for the same time, the distances traversed are as the speeds. And although the speed in arc DF is not uniform, and neither is the speed in arc EG uniform, since in neither arc does the planet remain a constant distance from the Sun (the cause of the motion), nevertheless, the smaller the amount of time we take, the smaller these arcs; hence the smaller the diversity of distances from the sun in each; hence the nearer the speed in each comes to being uniform. Taking them small enough, then, we treat them as uniform, and they take place in equal times, and the distances they accomplish are the arcs themselves,

So speed of planet in arc DF : speed in arc EG = arc DF : arc EG

Now, the smaller we take these arcs, and the nearer we are to D and E, the less those arcs differ from the chords DF, EG. Hence we get as near as you like to having the proportion:

speed in arc DF : speed in arc EG = DF : EG

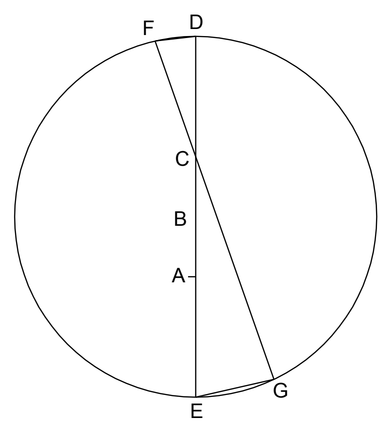
But also, as F is taken nearer to D, and G nearer to E, CF becomes as close as we like to being equal to CD, and CG becomes as close as we like to being equal to CE. So these triangles are becoming as nearly isosceles as we like. Therefore we get as near as you like to having the proportion:

FD : DC = GE : EC

or DF : EG = DC : EC [alternating]

And so the following proportion becomes as close to true as we like, as F and G are taken nearer and nearer to aphelion and perihelion respectively:

speed in arc DF : speed in arc EG = DC : EC



But, thanks to the bisected eccentricity,

DC = EA
and EC = DA

So, as near as you please, we approach this proportion as we take tinier arcs near aphelion and perihelion:

speed in arc DF : speed in arc EG = EA : DA

Q.E.D.

That is only a single pair of locations for the planet, namely at D and E, but Kepler seems to take it as a general rule that the speed of the planet is inversely as its distance from the sun. (In Newton's *Principia*, in the corollaries to the first Proposition of Book 1, we find that the true rule for the ratios of the speeds of the planet at different locations in its orbit is somewhat different from what Kepler supposes.)

QUESTION: The argument above seems to suppose that the triangles CDF and CGE get as near to being similar as we please, as we shrink the arcs accomplished in equal times. But aren't those triangles *exactly* similar?

Yes, they are, but in the opposite way we need them to be similar for this proof! Since ∠FDE stands on F and E, and ∠FGE also stands on F and E, therefore,

∠FDE = ∠FGE

and ∠GFD = ∠DEG for the same sort of reason.

So △CDF is similar to △CGE.

But their similarity, given which angles in them are equal, gives us the proportion:

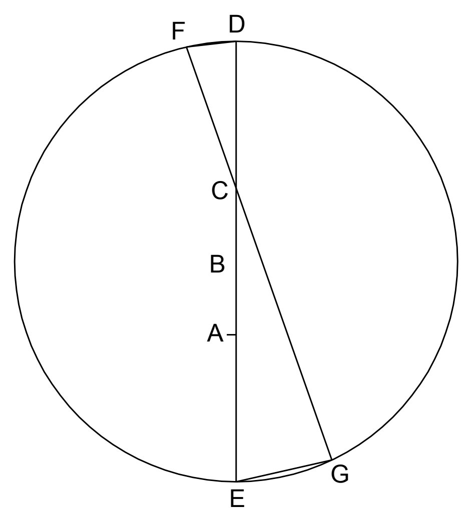
FD : DC = GE : GC

whereas we want to say

FD : DC = GE : EC

So we need to argue from the fact that GC and EC differ from each other as little as we like, as we approach perihelion and aphelion.

QUESTION: Can we make the same sort of argument for any other two points on the orbit which lie on a straight line through the equant-point? No, we cannot. Nor is Kepler's rule true for other pairs of points. Aphelion and perihelion are the only two places where his rule is exactly correct (and coincides with Newton's rule).



CHAPTER 33

THE POWER THAT MOVES THE PLANETS
RESIDES IN THE BODY OF THE SUN

[1] CHANGE IN DISTANCE FROM THE CENTER IS THE CAUSE OF CHANGE IN SPEED.

Kepler begins this chapter with an "axiom" of the philosophy of nature:

If X and Y occur at the same time and in the same manner or measure,
then either (a) X is the cause of Y
or (b) Y is the cause of X
or (c) some third thing is the common cause of both X and Y.

But it is just so with the speed of a planet around the Sun (call this "X"), and with the inverse of its distance from the Sun (call this "Y"). So either one of these is the cause of the other, or some third thing is the cause of both.

But, says Kepler, it is not possible to imagine some third thing concurrent with both the speed of the planet and its distance from the sun which could be the cause of both. Therefore NOT (c).

So one of these must be the cause of the other. But which is cause of which?

Well, "distance" is prior to "motion" and "speed," since you can have distance without motion, and conceive it without motion, but motion and speed cannot be, or be conceived, without distance. Therefore, thanks to the priority of Y to X, or of distances to motions, it is impossible for X to be cause of Y. Therefore NOT (a).

And therefore (b) is the truth in this case. That is, the solar distances of a planet are the reason for its various speeds.

[2] THE CAUSE OF THE SPEED IS AT ONE ENDPOINT OF THE DISTANCE, i.e. AT THE END THE PLANET IS NOT ON.

Taking it as evident from what has been said so far that the cause of the variation in a planet's speed must lie at one end or the other of the Sun-planet line (since the sun and the planet are the only two relevant bodies along that line, and mathematical points are incapable of being physical causes), Kepler will now argue that the cause must not lie at the planet's end, but at the Sun's:

(a) Since the body is not made heavier or lighter by approaching or receding (Whoops! That is not quite true! But he takes it as evident, presumably because the body's nature is not altered by a change of place),

(b) Since the body would get tired by speeding itself up and slowing itself down,

(c) Since it has no means of moving itself, i.e. no wings, no solid orbs.

[3] THE CAUSE OF THE SPEED IS AT THE CENTER OF THE WORLD.

This is the alternative. And as soon as we embrace it, the idea suggests itself to us that the motive cause uses the center of the world like a fulcrum, and moves the planets as if by levers, so the further they are, the harder it is to move them (just as with a see-saw or a door or any other kind of lever). And that is the reason they slow down at greater distances from that point.

[4] THE SUN IS THE CENTER OF THE PLANETARY SYSTEM.

So what is this "center"? Is it just a mathematical point, with no power at all? Is it the Earth, as for Ptolemy? Is it the Mean Sun? Is it partly the Mean Sun, and partly the Earth, as for Brahe?

He has already given probable arguments (e.g. in the Introduction) that it is the body of the Sun.

He asserts that the lines of apsides of the planets pass through the body of the Sun. Hence he concludes that:

[5] THE MOTIVE POWER IS IN THE SUN.

[6] THE SUN IS THE CENTER OF THE WORLD, AND DOES NOT MOVE FROM PLACE TO PLACE.

He gives us another "either-or" (this whole chapter is a marvelous example of various logical procedures):

EITHER a motive power in the Sun moves Earth & all planets,

OR a motive power in the Earth moves the Sun & all planets.

Brahe destroyed the orbs, so these cannot be appealed to as causes of motion for the planets.

And Kepler has proved there is an equant in the theory of the Earth's motion, or else of the Sun's motion, depending on your view of the matter. But that means the Earth moves faster when it is near the Sun, OR ELSE THE SUN MOVES FASTER WHEN NEAR THE EARTH. And therefore, thinking as a physicist (and employing the Axiom used earlier), we must say either that the Sun moves the Earth, or that the Earth moves the Sun. The idea that the Earth moves the Sun is absurd. Therefore the Sun moves the Earth.

Kepler realizes he must, by his principles, admit that the Earth moves the Moon, and that is hard for people to believe. He thinks of the Moon and Earth as kindred bodies, since he sees that the tides are caused by the Earth's seas being attracted to the Moon.

[7] THE KINSHIP OF THE SOLAR MOTIVE POWER WITH LIGHT.

If we have two concentric circles, radii R and r, with the motive power of the sun M at the common center, then, says Kepler, we have shown that:

power of M at distance R : power of M at distance r = r : R

But that is the same rule for "candle power," or illuminating power in a light source. So this power weakens with distance by the same rule that light does, and therefore the motive power of the sun appears to be akin to light.

(Kepler does not quite have the rule by which the force on the planets keeping them in their orbits varies with distance from the Sun. It is not inversely as the distance, but inversely as the square of the distance. Nonetheless, the kind of argument he is now making is remarkable, and Newton himself will use one very like it in *Principia* Bk. 3 Proposition 4, to establish that the force keeping the moon in its orbit is its weight toward earth. Newton argues there that the force keeping the moon in its orbit has the same quantitative properties as the force drawing a stone to earth, and therefore it is the same type of force. Here, Kepler is arguing that the force keeping planets in their orbits obeys the same quantitative rule as light, and therefore that force must be of much the same nature as light.)

Nonetheless, he says it is NOT the Sun's light itself which is its motive power, since it is hindered by the opaque, and so darkness would slow bodies down, e.g. during eclipses, which does not happen. Again, light spreads from the Sun in straight lines, but the motion of the planets is in circles.

NOTE: The weakening of the power as it spreads out (like a circle of light from a flashlight which is made bigger by receding from the wall) explains not only why one planet speeds up and slows down, but also why planets further from the Sun have longer periods. This is a First Law of planetary motion which Copernicus thinks very important—but he is unable to give a reason for it, given his physics.

[8] THE MOTIVE POWER IS AN IMMATERIAL SPECIES OF THE SOLAR BODY.

Kepler speaks of the solar influence as an "immaterial species." It is "immaterial" because, like light, nothing of it is lost in transmission from a smaller circle to a greater one (it is just less intense).

KEPLER

DAY 41

THINKING PHYSICALLY ABOUT PLANETARY MOTION
(Chapters 34, 38, 39)

CHAPTER 34

THE SUN IS A MAGNETIC BODY, AND ROTATES IN ITS SPACE

In the chapters we are now drawing from, Kepler is thinking in a serious way for the first time in human thought about the *physics* of celestial motions. What sorts of causes produce those motions? How do they operate?

He observes that the only way the “species” flowing from the Sun could *move* the planets is by whirling around (a thought later imitated by Descartes’ theory of “vortices”, which Newton disproves in many ways). And the Sun must spin to cause this motion in the “species” emanating from it. The poles of the Sun’s rotation, obviously, are the poles of the zodiac, since the motions of the planets are caused by the rotation of the Sun, and they move (basically) through the zodiac. Therefore the Sun is the *natural cause* of the zodiac, says Kepler.

Also, since the planets don’t move with equal speeds, therefore they lag behind the whirl which follows the speed of the Sun. Therefore the planets are material things, and are *inclined to rest*. (Note that this is not quite Newton’s idea of inertia, since he is not saying that they tend to keep going in uniform motion in straight lines if left alone. Rather, the planets are inclined to slow down and stop.)

Kepler goes on to observe that since

$$\frac{sun's\ radius}{radius\ of\ mercury's\ orbit} = \frac{earth's\ radius}{radius\ of\ moon's\ orbit}$$

therefore it is plausible that

$$\frac{time\ of\ sun's\ rotation}{time\ of\ mercury's\ orbit} = \frac{time\ of\ earth's\ rotation}{time\ of\ moon's\ orbit}$$

He works from the proportion:

Earth : Moon = Sun : Planets

The Earth rotates in the same direction that the Moon orbits it, and it does rotate faster than the Moon orbits it. So that gives us a way of calculating the (probable) period of the Sun's rotation. Kepler calculates that the Sun rotates once in about three days. (Later, by means of Sunspots, it will be possible for astronomers to observe the true period of the Sun's rotation.)

Kepler observes that magnets weaken with distance, and the Earth is a magnet (as he learned from Gilbert). So, given the similarities, he concludes that the Sun and the Earth are magnets and move bodies by magnetism. The Sun seems to influence Earth from a distance, and the Sun is a large celestial body like Earth, and we know that Earth is a magnet. And the best precedent in our experience for action between bodies at distances from each other is magnetism. (It would be interesting to ask him why he did not consider weight!)

Although he is wrong on some points, Kepler is right in a general way. The Sun does rotate, and in the same direction as the planets orbit it, although its period is about once in 27 days (it is a bit faster in the equatorial regions, 26 days, and slower around latitudes of 60°, i.e. 31 days, so the rotation is "differential"). Also, it is a huge magnet, a giant dynamo. On the Sun we observe Sunspots, pores, faculae, regions of intense magnetic field. The field is generated below the atmosphere, within the enormous rotating mass of gas (2×10^{30} kg). As they move, electrons and protons create an electric current which induces a magnetic field. Also, because of the Sun's rotation, the magnetic field's force lines rooted in the solar corona form an Archimedes spiral around the Sun, radiating out all through the solar system.

He is also right to try to assign a familiar force to the influence which the Sun has over the planets. But he is mistaken in thinking that the Sun's magnetism is the force by which it holds the planets in their orbits.

CHAPTER 38

BESIDES THE COMMON MOTIVE FORCE OF THE SUN, THE PLANETS ARE ENDOWED WITH AN INHERENT FORCE, AND THE MOTION OF EACH OF THEM IS COMPOUNDED OF THE TWO CAUSES

Since the Sun's power is uniform, why do planets circle it *eccentrically*? In this chapter, Kepler is wondering about that question. What is the natural cause of the eccentricity of the planetary orbits? He suggests that the reason for their eccentricity is that they are at varying distances from the Sun, and therefore they encounter different intensities of its power, or its "immaterial species," which are moving around it whirlpool-style. The nearer they are to the Sun, the more intensely are they subject to this sideways influence, and the faster they move in longitude. The further they are from the Sun, the less intensely are they subject to the same influence, and the slower they move.

But why is a planet at varying distances from the Sun? He supposes this is to some extent because of the Sun, but also to some extent because each planet has its own "motive power" or "*vis insita*." He compares the Sun's influence to a torrent of water, and the planet's "*vis insita*" or inherent force to a rudder or oar which it might use to resist the simple flow of the solar species and thereby bring itself now closer to, now further from, the Sun. So there is some tendency in the planets to reciprocate, to move back and forth, along the line joining them to the Sun.

This idea of a ferryman, a pilot in a ship, a steersman (cf. "*gubernator*") suggests that the planet might have a mind of its own, or is self-directive. Certainly it was the view of the ancients, such as Plato and Aristotle, that the heavenly bodies have intelligence and self-directed motion. While Kepler will consider this view, in the end he will reject it.

CHAPTER 39

BY WHAT PATH AND BY WHAT MEANS DO THE POWERS SEATED IN THE PLANETS NEED TO MOVE THEM IN ORDER TO PRODUCE A PLANETARY ORBIT (THROUGH THE AETHEREAL AIR) THAT IS CIRCULAR, AS IT IS COMMONLY THOUGHT TO BE

In this chapter, Kepler lays down six physical axioms which he says are "of great certainty." They are the following:

1. A planet is sluggish, and inclined to rest.

2. A planet is moved in longitude by the Sun's power.

3. If a planet were a constant distance from the Sun, then it would move in a circle.
(This sounds almost like a tautology! But that is not quite so. First, the predicate adds to the subject that the orbit is in one plane, not some wild line on a sphere. Second, he means that there is a cause moving the planet in a circle, namely the Sun's spinning and its whirling immaterial species, and this cause does not draw the planets in toward the Sun, nor move them away from it. The planet's varying solar distances are therefore due mainly to some other cause, namely the planet's own inherent force.)

4. If the same planet orbits the Sun first at distance (or radius) A, then at radius B, then the periodic times will be as the squares on those radii.

(He seems to be thinking as follows. The further out you are, the *greater the circumference of your orbit*, directly as the radii. But also, the further out you are, *the slower your speed*, directly as the radii, as he said before. Hence the further out you are, the longer your period, as the squares of the radii. This rule, as stated here, is not quite correct, and is contrary to his own Third Law of planetary motion, namely that the periodic times of planets in different orbits are not as the squares of the radii, but rather the squares of the periods are as the cubes on the mean radii. But apparently he arrived at his Third Law of planetary motion from beginning with imperfect thoughts such as this one.)

5. The power of the planet is not enough to make it move in longitude or to make circles, "*Since it lacks feet, wings, and feathers by which it might press upon the aethereal air*." (William H. Donahue translation.)

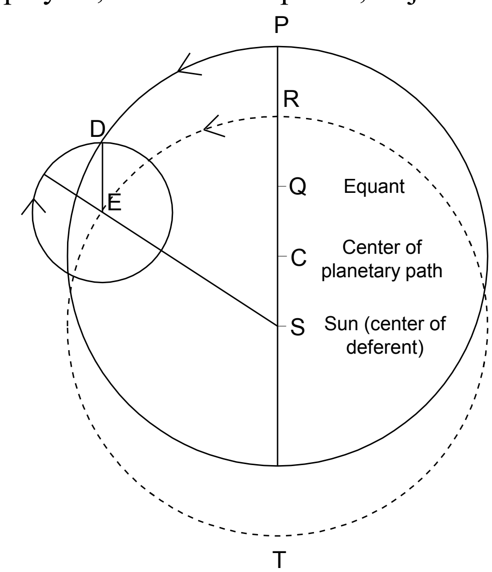
6. The planet approaches and recedes from the Sun by the power which is proper to the planet. (This is half true, since it is the weight of the planet together with the inertia of its mass which is responsible for its orbit.)

Now, adopting these physical axioms, Kepler wants to begin looking at what geometrical figures they might trace out in our understanding of any particular planetary orbit.

It is at this point that Kepler rather surprisingly re-introduces the idea of an EPICYCLE. But he is re-introducing it basically to destroy it. His main point seems to be this: IF there were solid orbs, then the planet would have a physical reason to rotate about the center of that solid, and hence about the center of the epicycle. BUT THERE ARE NO SOLID ORBS, and therefore all the points become imaginary, "incorporeal," and we cannot think of any *physical* causes for motion around them.

But Kepler will continue to talk about planetary epicycles all the way to the end of the book, even while establishing that the planets move on perfect ellipses, and even after showing that epicycles are physically absurd notions. This is because they are useful devices for calculating planetary solar distances.

Way back in Ch. 2 (which we skipped over), Kepler introduced a certain epicyclic hypothesis. In this hypothesis, ED, the radius of the epicycle with center E, is always parallel to itself in an opposite-direction epicycle, and hence moves around E with the same speed (though in the opposite direction) with which its epicycle moves around S (the Sun)—and, by the way, since that speed is non-uniform around S, so too the planet (whose line to E is always parallel to itself) moves around N non-uniformly, and so there is an equant-point inside the epicycle itself (which is why Kepler says, earlier in the book, that the solar theory is built into every planetary theory; the epicycle, for an outer planet, is just a replica of Earth's orbit). The epicycle's radius, DE, is equal to SC, the eccentricity. He proves this epicyclic hypothesis is equivalent to what I have called the CRUDE hypothesis, the simple eccentric orbit (the solid line), but we can see it right here for ourselves. It is just like Ptolemy's proof of the equivalence of the two solar hypotheses (simple eccentric or concentric deferent with an epicycle having a radius equal to the eccentricity, etc.). Just keep DE parallel to itself, E always on the dotted deferent, and the planet will trace out the solid orbit.

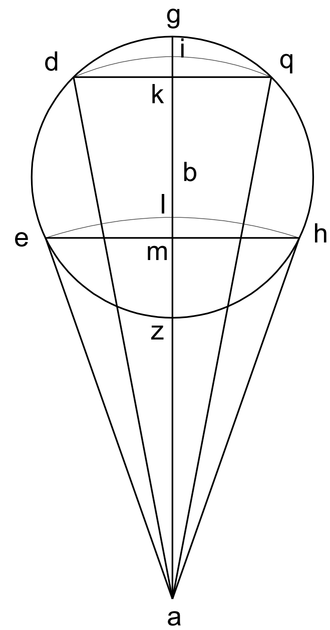


Kepler's point here is that this epicycle, while GEOMETRICALLY equivalent to the CRUDE hypothesis, is NOT PHYSICALLY EQUIVALENT. It is in fact physically absurd. This observation brings the specific benefit of destroying epicycles for good (except as calculation tools, in which form they arise again in Ch. 56), and also the more general benefit of teaching us that geometrical equivalence does not mean EQUAL PLAUSIBILITY. When we get the geometry and the physics to work together, THEN we are on the path to truth!

An epicycle was one way to explain why a planet was different distances from the Sun at different times. The planet sat on a solid crystalline sphere which rotated, and hence brought the planet now closer to, now further from, the Sun. But if we do away with such spheres, how are we to understand the physics underlying the planet's variable distance from the Sun?

Kepler's idea is that the planet "reciprocates" or moves back and forth along the line joining it to the Sun, and does so by its own power, and in accord with a rule that is equivalent to motion on an epicycle. This is why he keeps bringing epicycles back into the picture, although he does not believe in them as physical things (i.e. actual rotating spheres).

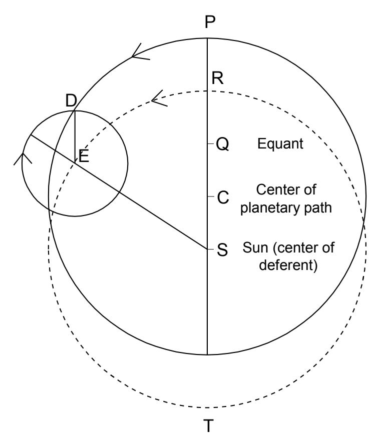
At the moment, however, he is examining the physical viability of epicycles. He is imagining the planetary "mind" cutting off what it deems to be appropriate solar distances like ai, and using these to determine where it should be on its epicycle. "Ah!" the planet might say to itself, "I should be at q on my epicycle now, since ai = aq." (This is why he brings up RECIPROCATION. The "relevant point" on AN keeps shifting back and forth in accord with distances from the Sun.)



But if insist on all this, he says we get FIVE ABSURDITIES:

ABSURDITY 1: The planet is making motion on its epicycle by its own power—and this is contrary to Axiom 5, since it lacks feet, wings, etc. Since it lacks moving parts, no physical cause can be assigned for such a motion around some invisible and intangible and inactive point b.

ABSURDITY 2: Returning to our other diagram, D moves about C with the same angular velocity as E moves about S, as we saw. Therefore the motion of D about C and the motion of E about S intensify and remit at the same times (note C is the geometric center, not the equant, so the epicycle does not move uniformly around C, nor does the planet). Therefore, since D speeds up and slows down around the Sun, depending on its distance, it must be that E speeds up and slows down around S *even though it is always the same distance from the Sun*. But E should move with uniform circular speed around S if it is always the same distance away from the Sun, since at a fixed distance from the Sun, the Sun's immaterial species moves at a fixed rate or with a fixed strength.



ABSURDITY 3: The Sun's motive rays move faster than the planets (see Ch. 34), yet we are now supposed to imagine a power-ray from the Sun, SE, always through the epicycle's center. Nor can we say the Sun's power-rays move through the center of the epicycle, E, faster than E itself orbits, since E cannot be a "sluggish point", being incorporeal. The Sun, in pushing the center of the epicycle, is pushing a ghost.

ABSURDITY 4: SE must speed up and slow down around S, since CD does around C, and SEDC is always a parallelogram, as we saw. And yet E is a constant distance from the Sun (this is very similar to absurdity 2).

ABSURDITY 5: We have to think the planet orbits about an imaginary point E and in the opposite direction of the longitudinal motion. How can physics explain that?

FORGET THE EPICYCLE, AND STILL . . .

Kepler now imagines someone saying "Forget the epicycle! That is not the cause of the epicycle-like movement. The cause is instead the mind of the planet, which directs its own motion by a very simple rule. Just let ED stay always parallel, and let E stay equidistant from S." But on that supposition, he says, we still have absurdities:

ABSURDITY 1: Even if a planet has a mind, how does it keep track of point E, where nothing is? Even if calculated from the location of C, C is also nothing but a thought-product. (And we have to give it a mind, because there is no physical cause at E for the motion of the planet around it.)

ABSURDITY 2: It is silly to think the planets know the tables of the planetary motions.

ABSURDITY 3: The infinite is unknowable.

The continuous change in the proper distances of the planet, such as SD, would mean that the planet must consider an infinity of such distances in any length of time.

Kepler does not think that it is possible to account for this motion of the planet by a mind or by any "*vis insita.*" Instead, the motion of the planet which produces its various distances from the Sun must somehow result from the concerted action of the Sun and planet.

KEPLER

DAY 42

Chapter 40

AN IMPERFECT METHOD FOR COMPUTING THE EQUATIONS FROM THE PHYSICAL HYPOTHESIS, WHICH NONETHELESS SUFFICES FOR THE THEORY OF THE SUN OR EARTH

KEPLER'S THREE LAWS OF PLANETARY MOTION.

Kepler is famous for having discovered the three laws of planetary motion. All three are of great significance for astrophysics, and Newton reasons from all three of them to the law of universal gravitation. These laws are as follows:

Law 1. Each planet moves on an ellipse which has the Sun at one focus.

Law 2. The line from each planet to the Sun sweeps out equal areas in equal times.

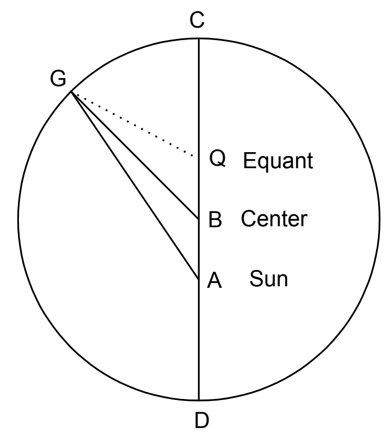
Law 3. Comparing any two planetary orbits, the squares of their periods are as the cubes of their mean distances from the Sun.

Law 1 will be established by the end of the *Astronomia Nova*—at any rate, in the case of Mars (and it may be done similarly with the remaining planets). Law 3 is stated in Kepler's *Epitome of Copernican Astronomy* 4.3, and in other places besides, but we will not concern ourselves with establishing that particular law. But in Chapter 40 of *Astronomia Nova*, which we are now considering, Kepler establishes Law 2, although almost as an aside—it is approached as a means for computing "the physical part of the equation," i.e. the angle which is traversed around the equant-point in a given time. Kepler's second law is very significant, however, not only as a link to Newton's physics of centripetal forces, but also as a break with past thinkers. The ancient astronomers believed in some kind of uniformity of motion underlying the apparent irregularity in the movements of the planets. Modern astrophysicists agree with that general premise. But the ancient astronomers sought this regularity in perfectly uniform circular motion, whereas for modern astrophysics that would be a very special and particular case. More often, the celestial bodies do not sweep out equal ANGLES in equal times around any point at all, but instead they sweep out equal AREAS in equal times around the center of the centripetal force producing their orbits. Uniform angular velocity has been replaced by uniform areal velocity.

WHAT IS KEPLER DOING?

In the present chapter, Kepler is concerned principally with Earth's orbit.

The SUMMARY for Ch. 40 makes plain what he is aiming at: "A method by which the physical part of the equation, that is, the elapsed time of a planet over any arc of the eccentric, may be found from the distances of the points of its arc from the Sun." The title reflects this, too.



Why does he want this? If we supply Q as the equant-point in Kepler's diagram, then ∠BGQ is the "physical part of the equation," and ∠GQC is a measure of the TIME. This will tell us where a planet should be at a given time, in longitude. What he wants to do is show that his physical theory produces *consequences for the TIME*, that is, it dictates—and dictates correctly—what the angles about Q should be at any given time.

So he needs to reason synthetically, to reason forward, from causes to effects, beginning from his physical hypotheses, to arrive at a way of calculating, from these, what the angles are around Q (or what the angles AGB, BGQ are, which will be as good as GQC).

Given the time that has elapsed since the planet was at C, we would have the angle CQG, and given the eccentricities (i.e. the values of AB and BQ in terms of the orbital radius) we could easily calculate, by trigonometric methods, the values of the angles BGQ and AGB. But that is not all Kepler is doing here.

THE STARTING POINT

In Ch. 32, Kepler showed that the speeds at aphelion and perihelion are inversely as the distances from the Sun. He claims to have shown also that this rule is very nearly true throughout the orbit (the real rule for the speeds at two locations on the orbit is that they are inversely as the perpendiculars dropped from the Sun to the tangents at those two points).

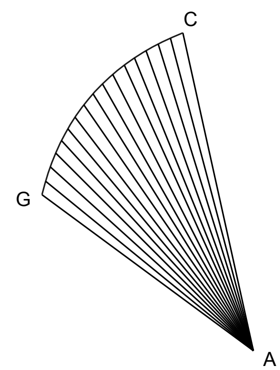
Since for Kepler the speeds are inversely as the distances from the Sun, and since the speeds are also inversely as the times they take to go an equal linear distance (the greater the uniform speed, the shorter the time for it to go a given distance), therefore *the times are directly as the distances from the Sun*.

That reasoning gets us from physical ideas to the result that the times are as the solar distances. But the problem with the distances is that there is an infinity of them in the eccentric orbit, and *no time* is spent at any one of those distances from the Sun.

So how do we find the time spent in any given arc of the eccentric orbit? We "ADD UP ALL THE DISTANCES" in very short, very many, arcs. The total distance, or rather AREA, will be proportional to the time spent sweeping out that area. This is the way we *stumble into Kepler's Second Law*. He is still so convinced of the importance of an equant, that he brushes past the second law as a pure means to an end! Really, it is the only thing that makes any physical sense (as Newton will later show), and which gives us a brand new way of clocking the motion of the planet. We now have an area-clock, as opposed to an angle-clock.

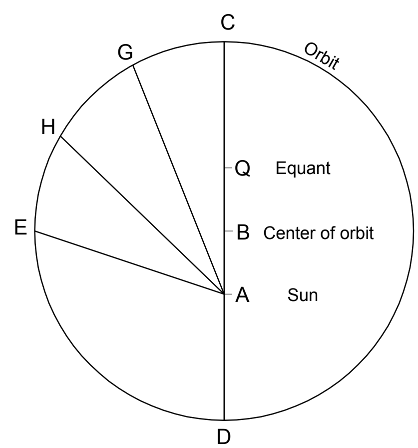
FIRST [HINTED] ARGUMENT THAT THE AREA IS PROPORTIONAL TO THE TIME (i.e. THE 2nd LAW), FOR THE SAKE OF CALCULATING, FROM PHYSICAL CAUSES, "THE EQUATIONS," i.e. THE ANGLES *AGB* etc.

Kepler hints at an argument where he says it "seemed to me ... that by computing the area CAH or CAE I would have the sum of the infinite distances in CH or CE."



First, without calculus (but in anticipation of it), we assume something which is strictly speaking impossible yet somehow intuitively persuasive: Area GAC is composed of an infinitude of distances drawn from A to arc GC. We assume that the area of the circular sector GAC is somehow the sum of an infinity of lines. And area HAG is likewise composed of an infinity of distances drawn from A to arc HG.

Thus $$\frac{\text{All the distances in arc HG}}{\text{All the distances in arc GC}} = \frac{\text{Area HAG}}{\text{Area GAC}}$$



But he said before (not quite correctly) that the distances from the Sun are inversely as the speeds of the planet when it is at those distances; but times are also inversely as speeds. Hence the times, or delays, of the planet (at points? well, at very tiny arcs!) are directly as the distances. So, since the times of the planet in each tiny point-arc are as the distances from the Sun (which distances it has during its time in each teensy arc), hence

$$\frac{\text{All the times in little arcs of HG}}{\text{All the times in little arcs of GC}} = \frac{\text{Area HAG}}{\text{Area GAC}}$$

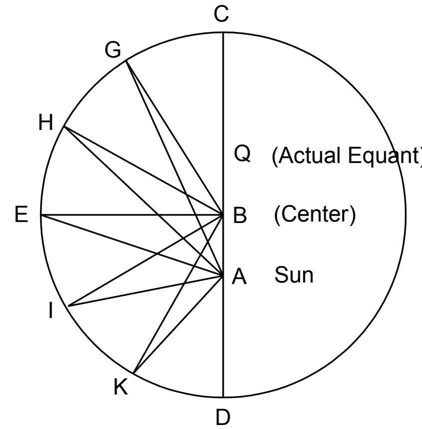
i.e. $$\frac{\text{Total time in arc HG}}{\text{Total time in arc GC}} = \frac{\text{Area HAG}}{\text{Area GAC}}$$

And that is Kepler's 2nd Law, i.e. that the areas swept out around A, the Sun, are as the times.

Kepler SUGGESTS this argument, but then seems to give ANOTHER ARGUMENT a little further on! (More on this in a moment.)

ONWARD TO THE ANGLES. . .

Kepler is interested in getting from this new time-rule to the rule for angles around the equant, i.e. a way of calculating them. He says "Thus the area CGA becomes a measure of the elapsed time or mean anomaly [movement] corresponding to the arc of the eccentric CG." That is, the planet is sweeping out equal AREAS in equal times around A just as it sweeps out equal ANGLES in equal times around Q, the equant. So the AREAS around A will be proportional to the ANGLES around Q.



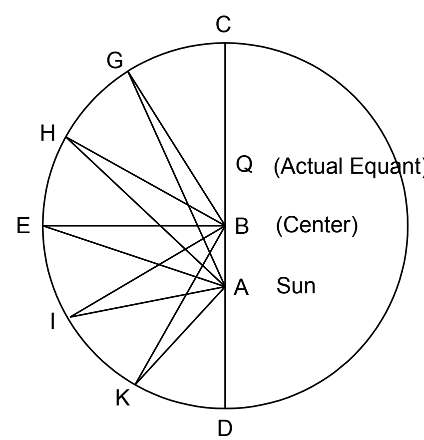
There is an occasion of confusion in Kepler's diagram—the arcs and angles around B are equal, but B is not the equant-point, and so the times of these equal arcs and angles are not equal. So why does he draw them? This is his SECOND ARGUMENT, or rather his first and only explicit one, and which is easily missed since it is so imperfectly expressed. It goes like this: All the sectors around B, standing on equal arcs, are equal, and *they are as the angles around B*. These are NOT swept out in equal times, since that would be true of the angles around Q. But if areas around B are as angles around B, that suggests that where angles are as times around one point (Q), perhaps areas are as times around another point (A). We are already expecting A to be a significant center of uniformity somehow. Also, because of the equant, it takes the planet more than half the time of the semicircle to cover half the semicircle, i.e. to complete arc CGHE. But look! The AREA standing on that arc from A is also greater than half the whole area of the semicircle! By a few suggestive hints like this, we draw the general conclusion that A is the right point around which the planet will sweep out areas proportional to times: "Therefore . . . as the area CDE is to half the periodic time . . . so are the areas CAG, CAH to the elapsed times on CG and CH."

Now the sector CGA has two components:

Area CGA = Sector CGB + $\triangle$BGA

so $\triangle$BGA = Area CAG – Sector CGB

But area CAG is proportional to the TIME, and therefore to the mean anomaly (i.e. to angular motion around the equant-point on the eccentric equant-circle), i.e. to $\angle$CQG.



And area CGB is proportional to the eccentric anomaly, i.e. to $\angle$CBG (sectors are as the angles, in a circle).

Hence the remainder, $\triangle$BGA, must be proportional to the difference between the angles CQG and CBG, i.e. to angle BGQ, or "the physical part of the equation."

So if we know the triangle BGA, i.e. its angles and its sides and hence also its area, then we know both (1) the "optical part" of the equation, i.e. $\angle$AGB, and also the "physical part" of the equation, i.e. $\angle$BGQ (since the area of $\triangle$BGA is proportional to that angle).

Hence Kepler concludes: "Thus the knowledge of this one triangle [i.e. △BGA] provides both parts of the equation" for the movement GAC.

And that concludes Kepler's correlation of his physical theory with the angular measurement of planetary motion.

AFTERNOTE. This is a perfect example of DISCOVERY-MODE thinking. It is finding the true from the true-enough-but-still-false! Look how many false things go into finding this true thing (the 2nd Law):

1. A false orbit (circle).
2. A false rule for the velocities (as inverses of distances from the Sun).
3. A false motive (to discover how to find angles around the equant, a point we have no general reason to believe exists).
4. All the false stuff about composing things of indivisibles. (Genuine methods of calculus do not require us to compose continuous things out of an infinity of indivisibles.)

NOTE: Kepler's 3rd Law, that the cubes of the major axes of planetary orbits are in the same ratio as the squares of the orbital periods, can be verified just by plugging in the numbers:

	Period in Days	Mean Solar Distance in Miles
Mercury	88	36,000,000
Venus	225	67,200,000
Earth	365	92,900,000
Mars	687	141,500,000
Jupiter	4333	483,300,000
Saturn	10759	885,200,000

KEPLER

DAY 43

THE DESTRUCTION OF THE CIRCLE; FINDING SOLAR DISTANCES FOR MARS

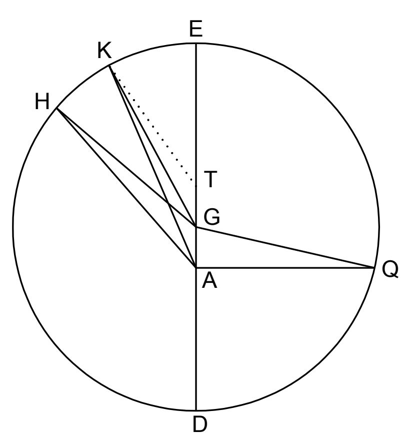
Chapters 44, 51

CHAPTER 44

THE PATH OF THE PLANET IS NOT A CIRCLE! NOT EVEN WITH RESPECT TO THE FIRST INEQUALITY ALONE EVEN IF YOU MENTALLY REMOVE THE BRAHEAN AND PTOLEMAIC COMPLEX OF SPIRALS RESULTING FROM THE SECOND INEQUALITY IN THOSE TWO AUTHORS

This is, at long last, the banishment of the circle as an adequate model for the orbit of Mars. Recall the C.W.A.B.E. hypothesis, the very simple hypothesis in which the orbit of Mars simply is a perfect Circle, With A Bisected Eccentricity (so that AG = GT, as required by certain observations of Ptolemy). We will now show that this circle cannot be the true orbit of Mars. But we don't, quite yet, see that the orbit is an ellipse. Kepler goes back to the drawing board, without preconceived ideas as to the shape of the path, and he goes through a false oval hypothesis (but let's not spend much energy on that!).

So the essential thing, here, is a negative result: NOT a circle.



He gets there like this:

FIRST ARGUMENT AGAINST THE CIRCLE (FROM LONGITUDES)

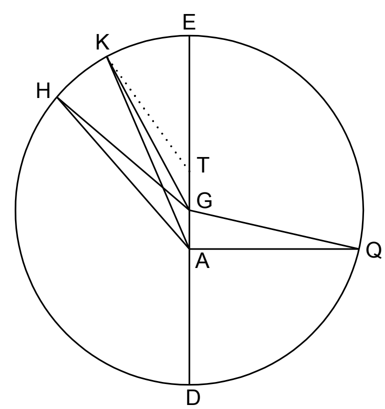
Take the planet in positions K, H, Q at known times, thus giving the angle from aphelion E (the position of the line of apsides EGAD was previously found with great accuracy).

Since we know the time, we also know the angle ETK (if T is the equant).

Hence	$\angle KTG = 180° - \angle ETK$	[so $\angle KTG$ is known]
but	KG is known	[radius of the eccentric]
and	GT is known	[= AG, bisected eccentricity]

and so, in $\triangle KTG$, we know two sides and a non-included angle, and so, by the Law of Sines:

$$KG : \sin(\angle KTG) = GT : \sin(\angle GKT)$$



so $\quad \angle GKT$ is known

thus $\quad \angle KGT = 180° - \angle GKT - \angle KTG$ is given

so $\quad \angle KGA = 180° - \angle KGT$ is given

so, by the Law of Cosines:

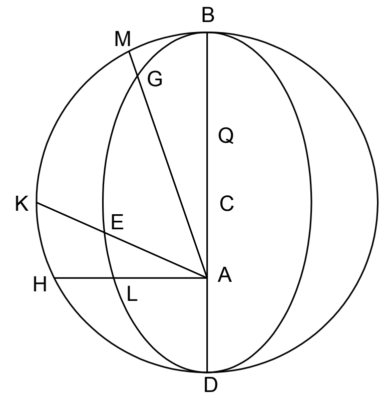
$$KA^2 = KG^2 + AG^2 - 2KG \cdot AG(\cos\angle KGA)$$

hence $\quad$ KA is given

Now repeat for HA and QA, assuming that KG = HG = QG = radius of eccentric orbit.
The resulting values of KA, HA, QA are noticeably off from other values found independently by more direct observations, namely “acronychal” observations, i.e. at solar opposition, when Earth is on the lines AK, AH, AQ, and so the longitudes of these lines are directly observed.

There are several assumptions, still, in this line of reasoning. We assume that there is a point around which the planet sweeps out equal angles in equal times—which might be false, and hence the falsehoods following from the reasoning above might derive from that assumption, rather than the assumption of a perfectly circular orbit. But he will return to this and shred the circle many times, independently of such assumptions.

SECOND ARGUMENT AGAINST THE CIRCLE (FROM TIMES)



His second argument to the same effect is from the times.

If the orbit is a circle, then

Time in arc BM : Time in KH = Area BMA : Area KHA

If the orbit is oval on the line of apsides, then

Time in arc BG : Time in EL = Area BGA : Area ELA

Now Area BGA : Area ELA > Area BMA : Area KHA

Therefore, if the orbit were an oval, the planet would spend more of its time in ∠BAG and less in ∠EAL than it does on the hypothesis of a circular orbit, and *the observations bear this out*. So the time is not distributed as in the circle, but more of it is spent close to B and D than would be the case in the circle.

Here Kepler observes that the times are accumulated at aphelion and perihelion "in much the same manner as if one were to squeeze A FAT-BELLIED SAUSAGE at its middle," then one would squash the ground meat from the belly out to the extremes, above and below the hand. That doesn't seem to be an image of the orbit itself, i.e. the path, but of the way the times are distributed, or would be distributed.

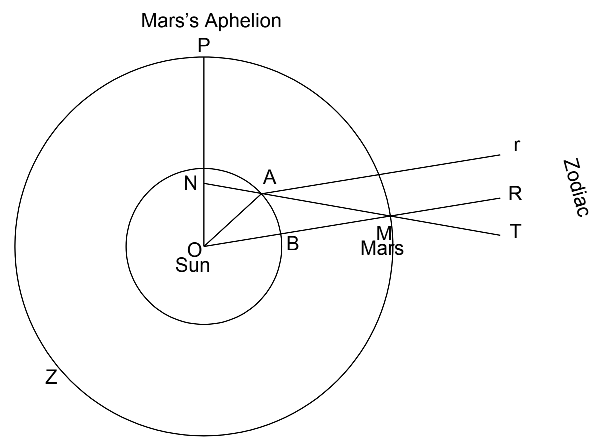
Note that this argument against the circularity of the orbit of Mars is independent of any assumption about an equant! It is based on his Second Law of Planetary Motion, which he discovered through physical considerations.

In Chapter 45 (which we will skip past) he pursues a BLIND ALLEY, i.e. a quasi-epicyclic theory, producing an ovoid path.

In Chapter 46 he finds another way to generate this hypothetical egg-shaped orbit. It has a sharp end and a blunt end, unlike an ellipse. But this eventually is rejected.

CHAPTER 51

DISTANCES OF MARS FROM THE SUN ARE EXPLORED AND COMPARED, AT AN EQUAL DISTANCE FROM APHELION ON EITHER SEMICIRCLE; AND AT THE SAME TIME THE TRUSTWORTHINESS OF THE VICARIOUS HYPOTHESIS IS EXPLORED



Kepler says Mars cannot be restrained by the oval (egg-shaped) orbit, so he calculates many more distances of Mars, without the presumption of a circular orbit.

He gets the true position of the line OM, where O is the Sun and M is Mars, from an acronychal observation, i.e. when Earth is right between O and M at B. Then he waits one full cycle of Martian motion, and he knows that Mars is again there on its orbit, at M, even if we don't observe it there, but are somewhere else on our orbit this time, such as A.

This eliminates the assumption of an equant-point, since we are waiting through a full cycle, and not worrying about how angles are distributed around some point.

GIVEN: We were once on OM, at B, and with an acronychal observation saw the true longitude of Mars, i.e. at R. One full Martian cycle later (to keep things simple), when we know Mars is in fact again exactly at point M, Earth is now at A, and we observe the planet at T in the zodiac.

FIND: OM, the distance of Mars from the Sun.

R is a definite star or place in the zodiac. So is T. And we observed Mars first at R, later at T. If we draw Ar parallel to MR, then they point to the same fixed star. So, not only is ∠rAT equal to ∠RMT, but r and R are the same point. So, when at A we observe Mars, which gives us the line of sight AT. And we also observe star R in the zodiac, where Mars appeared to us when we observed it from B, which gives us the line of sight Ar. Therefore ∠rAT is known and observed, i.e. ∠RMT, i.e. ∠AMO.

so ∠AMO is known

but ∠MAO is also known, since that is the angle between the Sun and Mars, with Earth at the vertex.

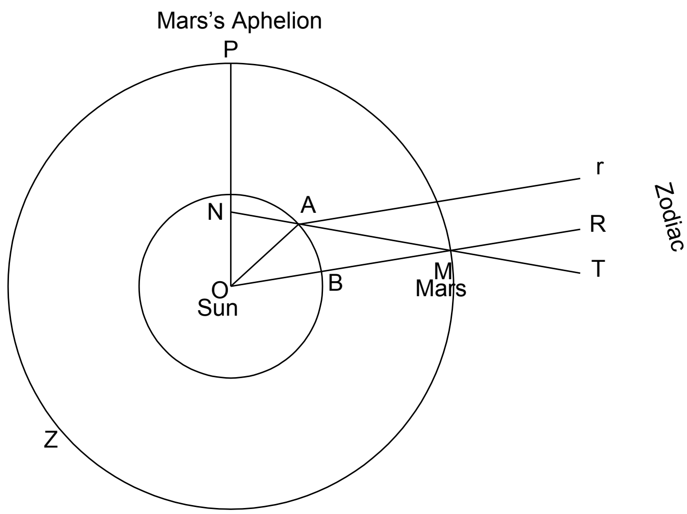
and OA is known (in units of Earth's mean distance from the Sun, given by the date and our solar theory, or theory of the Earth's movement)

But the Law of Sines says that

$$\sin(\angle AMO) : OA = \sin(\angle OAM) : OM$$

And we know all of these terms except OM.
Thus we find OM, the Sun-Mars distance.

Q.E.I.



Amassing the values of OM at various times will give us the pieces of a giant puzzle from which to construct the true shape of the orbit of Mars.

KEPLER

DAY 44

HUNTING FOR A WAY TO CONSTRUCT ALL THE TRUE SOLAR DISTANCES OF MARS

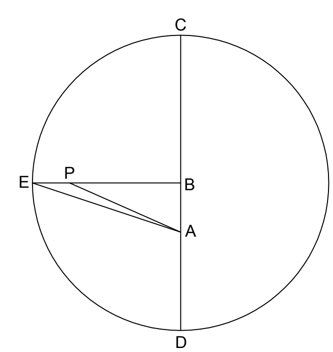
Chapter 56

In Chapter 55, Kepler shows that the "infallible observations" show both that the orbit is too skinny to be a circle, but also too fat to be the oval which he conceived in Ch. 45 (which we have skipped past). So it must be that the true orbit of Mars "takes a middle course" between the too-fat circle and the too-skinny oval.

Here in Chapter 56, Kepler hits upon a way to construct all the correct solar distances for Mars. This step is crucial, since it allows us to define the true orbit as the locus of points which all follow from the same construction, thus reducing the question of the shape of the orbit to a matter of pure geometry. Kepler had to find something *common* to all the true solar distances which Mars observes throughout its orbit in order to determine the shape of that orbit. This chapter does that for us.

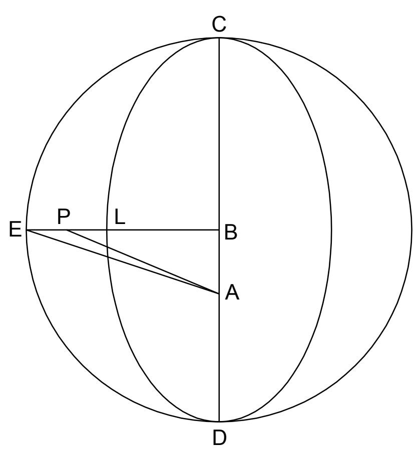
In Ch. 55, Kepler determined that the orbit of Mars is between the too-skinny oval and the too-fat circle—it is somewhere in the "lunule" which is the difference between these extremes. So he begins to hunt for a rule to determine by how much he has to "pull in" the orbit from the circle in order to make it correct. If we let CD be the line of apsides of the Martian orbit, B the midpoint, CED the circle on diameter CD which we know is close to the Martian orbit but a tad too fat, A the Sun, Kepler is asking himself this question: If AE is too long a solar distance to locate Mars correctly along the perpendicular BE, then what is the right solar distance? If Mars is actually at P instead of E (along BE drawn perpendicular to CD), what is the rule for determining that distance AP? He is particularly interested in examining this one case where BE is drawn at right angles from the midpoint, because that is where there is the *greatest difference* between the circular path and the true Martian path.

"The breadth of the lunule of ch. 46 above, born to us out of the opinion of chapter 45 which instructed us to cut it off from the semicircle—this breadth, I say, was found to be 858 units, of which the semidiameter of the circle is 100,000. But then, by two arguments . . . I concluded that the breadth of the lunule is to be taken as only half that, namely 429, or more correctly, 432, and in units of which the semidiameter of Mars is 152,350, nearly 660 . . ." (Ch. 56, William H. Donahue translation)



Kepler, like Ptolemy, will occasionally switch units on us and make proportional adjustments. In Ch. 42, he determined that the eccentricity, AB, for the Martian orbit, is 14,140 where the radius of Mars's orbit is 152,640 (and the radius of Earth's orbit is 100,000).

QUESTION: Why does Kepler here say "or more correctly, 432"? Because if EB is 152,350 (the value he uses in this chapter), and the eccentricity AB is 14,140 (from Ch. 42), then, adjusting this so that EB is our new reference length, EB = 100,000, AB = 9281. Then if the Martian orbit intersects EB at P, where angle BEA is given as 5° 18' (he calls this the "**greatest optical equation**")[1], and if we use his own technique which he is about to discover, i.e. if we assume that AP = EB = 100,000 (thus replacing the "secant" EA with the "radius" AP which is equal to EB, the radius), a couple uses of the Pythagorean Theorem easily prove that EP = 432. So then why does he say "429"? Because these are close enough that the number "429" at the end of the value of the secant for 5° 18' made an impression on him. More on this to come.



To keep things simple, let's stick with one set of numbers: Let the circle which approximates the orbit of Mars, and which shares its major axis as its diameter (CD, the Martian line of apsides), have a radius of 100,000 (so BD = 100,000). Let A be the Sun, CLD is the "oval" hypothesis of Ch. 45. Then where BC = 100,000, EL = 858. That is the "breadth of the lunule." But that breadth is too much, and would put Mars at L, whereas the Martian orbit actually passes through P, where EP is 429 (or, "more correctly," 432).

> "I therefore began to think of the causes and the manner by which a lunule of such a breadth might be cut off."

He is looking for a rule, in other words; a way of constructing, from the circular path, little adjustments in order to "pull in" the orbit to the true orbit of Mars.

> "While I was anxiously turning this thought over in my mind, reflecting that absolutely nothing was accomplished by chapter 45, and consequently my triumph over Mars was futile, quite by chance I hit upon the secant of the angle 5° 18', which is the measure of the greatest optical equation. And when I saw that this was 100,429, it was as if I were awakened from sleep to see a

[1] Actually, the greatest optical equation occurs, at least for the theoretical path, when EA is perpendicular to AB, as in Ptolemy's "greatest anomalistic difference." Nonetheless, Kepler seems to be talking about the case I am discussing here, when EB is perpendicular to AB, since the numbers work out, and since that does appear to be the simplest case. Perhaps he calls ∠AEB the "greatest optical equation" only because it is the optical equation which is a part of the greatest equation. That is, the total equation at point E, when BE is perpendicular to CD, seems to be the greatest angle, i.e. angle A-E-equant.

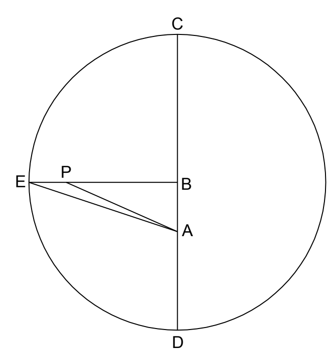
new light, and I began to reason thus. At the middle longitudes the lunule or shortening of the distances is greatest, and has the same magnitude as the excess of the secant of the greatest optical equation 100,429 over the radius 100,000. Therefore, if the radius is substituted for the secant at the middle longitude, this accomplishes what the observations suggest. And, [using] the diagram [of] Ch. 40, I have concluded generally that if you use HR instead of HA, VR instead of VA, and substitute EB for EA, and so on for all of them, the effect on all the eccentric positions will be the same as what was done here at the middle longitudes." (Ch.56, William H. Donahue translation)

Mark that: "Quite by chance." So we can't keep chance out of the discovery process!

What is Kepler talking about? What was his piece of luck? While trying to find a rule for determining the true "breadth of the lunule" (at all points) he ran across the secant of 5° 18', which is 100,429 (if the radius of the circle is 100,000). Those last three digits caught Kepler's eye: 429! That could not be by chance! 5° 18' is the "measure of the greatest optical equation" AND "429" is the correct breadth of the lunule at that point! (Actually, it is 432, but these were close enough to get him thinking.)

So the two numbers, 5° 18', and 429, caught his eye, because these two numbers are paired in two ways: (1) as an angle and the last digits of its secant, (2) as the "measure of the greatest optical equation" in the Martian orbit and the correct breadth of the lunule at that point.

The angle BPA is what he calls the "optical part of the equation," since one of its sides lies along our line of sight to Mars, namely when we make an "acronychal observation", i.e. during solar opposition, when Earth is along the line AP.



Kepler already had determined the value of EP when EB is at right angles to CD, the line of apsides. He also already realized that EA is the secant of the "greatest optical equation," since angle EBA is right. So what is new here? What is new is that

100429 – 429 = 100000

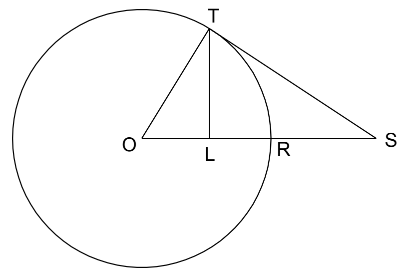
Now that would all collapse onto one straight line, of course, and not form a triangle. But EP is not 429 exactly. It is 432 exactly. Hence, even if PA were exactly 100000, these three lines EP, PA, EA would not collapse into one straight line, but would leave us with a very flat triangle, where EP is 432, EA is the secant of that, or 100429, SO WHAT WOULD AP HAVE TO BE? *Pretty close to 100,000!* That is, pretty close to the radius of the circle on the line of apsides as diameter. Do a little trigonometry, and it turns out it is *exactly* 100,000.

Wow! Could that possibly be a general rule for constructing point P? Let us ask it this way: *Of what general rule are lines AP and EB an instance?* In the next portion of Chapter

56, Kepler will attempt to describe the general rule, and he will get it half right, as we will see in Day 45.

QUICK VERIFICATION THAT SEC 5° 18' = 100429.

In our unit circle, let ∠ROT be 5° 18', i.e. 5.3°. Then OL, the cosine of this angle, is .995724698. Draw in tangent TS, forming secant OS.

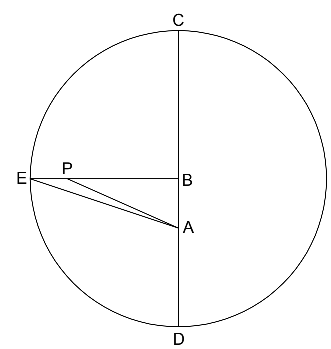


Now OS : OT = OT : OL
or OS : 1 = 1 : OL
so OS = 1/OL = 1/.995724698
so OS = 1.00429
or OS = 100429, where OT = 100000.

Or, again, looking in the diagram of the orbit,

cos∠E = EB/EA
so sec∠E = EA/EB (the reciprocal of the cosine, as we just saw)
so if EB = 1
then EA = sec∠E

So Kepler says EA is the wrong length for constructing the true distance to Mars. What we need is AP, which turns out to be equal to EB. "Therefore if the radius is substituted for the secant at the middle longitude, this accomplishes what the observations suggest."

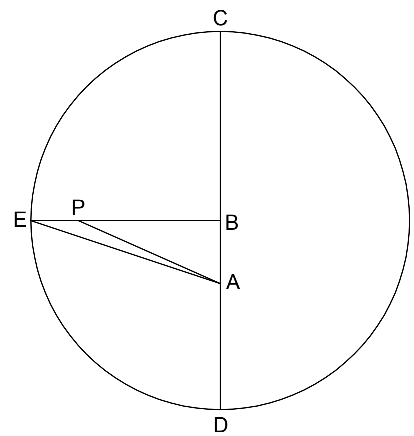


QUICK VERIFICATION OF THE GEOMETRY IN THE ORBIT.
Let EB = 152,350 (as Kepler says in this chapter)
Let AB = 14,140 (Ch. 42)
Or, adjusting
Let EB = 100,000
so AB = 9281.
Now construct AP = EB = 100,000,
and suppose ∠BEA = 5.3°

Then $EA = \surd(EB^2 + AB^2) = \surd(100000^2 + 9281^2) = 100429$
and $BP = \surd(AP^2 - AB^2) = \surd(100000^2 - 9281^2) = 99568$

so $EP = EB - BP = 100{,}000 - 99568 = 432$

so $EP = 432$, just like Kepler says.



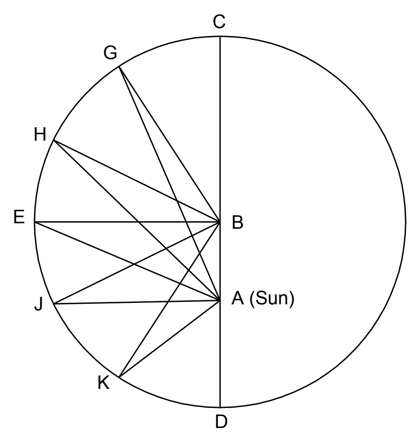
KEPLER

DAY 45

CONSTRUCTING THE SOLAR DISTANCES BY THE EPICYCLE; KEPLER GETS THE CONSTRUCTION RULE WRONG

Chapter 56 (Continued)

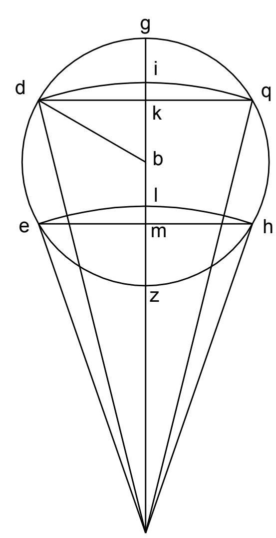
In the remainder of the drama of Chapter 56, Kepler brings back the equivalent epicycle of Ch. 39 used to construct points on the orbit. He is saying that what we have just discovered, our new rule, tells us that in the epicyclic equivalent we should use *ak* as a solar distance instead of *ad* (or *ai*), and *am* instead of *ae*.



This may not seem very important. It might even seem like a weird sidetrack into which the mind of Kepler keeps slipping, and to which he returns now in passing. But it keeps coming back, all the way to the end of the book! We will look into this more when we come to Ch. 58, where the "puff-cheeked orbit," his last dead-end, comes up, and where this epicycle comes up again.

The construction is this:

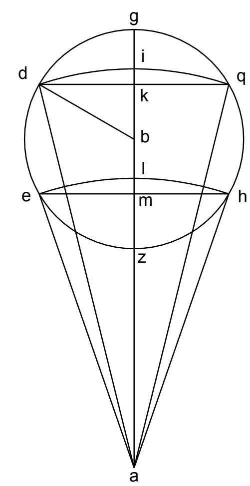
Let CGHEJKD be a perfect circle on the Martian line of apsides CD as its diameter. So this is the circle formerly believed to be the Martian orbit. A is the Sun, B is the center of the circle, and the points on the circle are taken at equal arcs.



Let the epicycle have a radius bd which is equal to the eccentricity AB. Let the deferent be a circle equal to CGHEJKD, but with A as center. Let Mars move on the epicycle in the direction opposed to that in which the epicycle moves on the deferent, and with the line bd (joining the epicyclic center to Mars itself) always parallel to the line of apsides. (This will result from making the pace of the planet on the epicycle match that of the epicycle about the deferent, which means there must be an equant-point inside the epicycle itself, since the epicycle does not sweep out equal angles in equal times around the center of the deferent.) We discussed this construction before, back in Day 41, where we saw that this epicyclic motion is exactly equivalent to Mars moving on circle CGHEJKD itself.

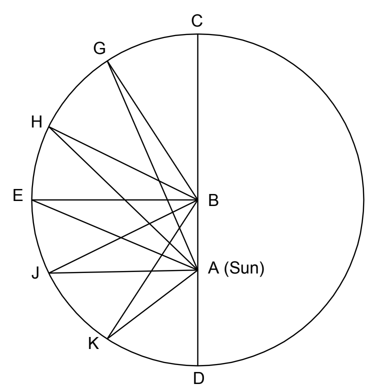
Recall that the way Kepler imagined the planet "thinking," was like this: "*I have to describe arcs on my epicycle that are always similar to the arcs which my epicycle is describing on the deferent. For instance, if the epicycle is at H, so that it has gone through arc CH, then I must be at q, such that arc gq is similar to CH. That is the only way I can produce the circular path which will make Kepler happy in Ch. 39. But, because he is thinking like a physicist, and is trying to give my epicyclic motion a physical significance, Kepler insists that I determine my rate on the epicycle by 'reciprocating' through various distances on the very line of apsides of my epicycle itself. I am supposed to 'feel' the distance which I ought to have from the Sun by its different tugs on my epicycle. So if I feel the tug on it of such an intensity which corresponds to point i on my epicycle's diameter, that tells me I should be at q on my epicycle, where aq = ai. Is this weird? Of course it is. But Kepler's mind is inscrutable.*"

All right, but the circular path is wrong, as we have seen. So now we need a new way for the thoughtful planet to construct its proper locations, and generate the right orbit, rather than the circular one. We need the rule, in other words, that will be equivalent to the new "replace-the-secant-with-the-radius" rule.



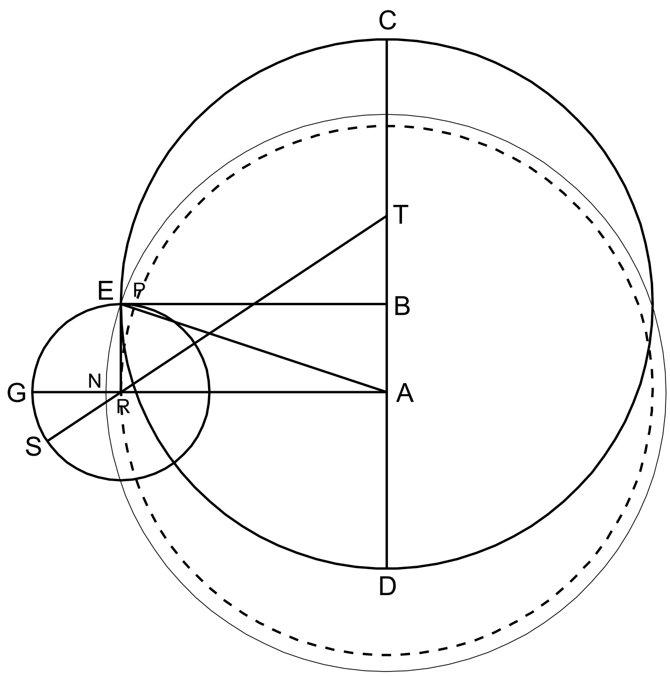
Formerly, we would have said that when the epicycle is at H, then Mars should be at q, where arc gq is similar to arc CH, and aq is equal to AH. In fact, aq *is* AH when we place the epicycle in the diagram of the supposed orbit CGHEJKD.

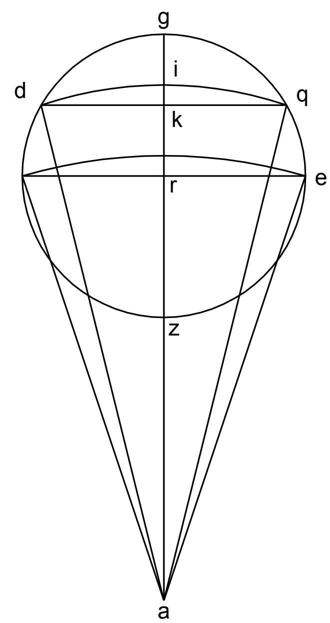
But that lands us right on the circle, which is not the true orbit. So instead, we don't want aq, but something slightly smaller. What is it? It is ak.



Let's see this new equivalence in a single example, and an easy one: the case where the planet is at P, along EB, where EB is perpendicular to CD the line of apsides. This is the case we were talking about in Day 44, with the secant value of AE being equal to 100,429 and so on.

Let T be the equant-point. Circle CD is the old theoretical orbit, B its center, A the Sun. The dotted line is the deferent, center A, equal to circle CD. We let the planet go to E such that ∠EBC = 90°, to examine our special case. From E we draw a line ER parallel to CD and equal to eccentricity AB. Hence R is the center of the epicycle. Joining TR through to S on the epicycle, we see that since the epicycle has gone through ∠CTR in regular motion, the star must have gone through an equal angle, ∠SRE, since it was up at apogee at C, but in the opposite direction. And since EBC is a right angle, ABER is a rectangle. Draw a circle of center A, radius AE, intersecting the line from the Sun through R (i.e. line ARG) at point N. According to the old "reciprocation" rule, then, when the epicycle is at R, the planet is supposed to "sense" the propriety of length AN, and from that calculate or react appropriately, so that it is at E on its epicycle, with AE = AN.





But the planet is not really at E, but at P, we now know. So instead of taking

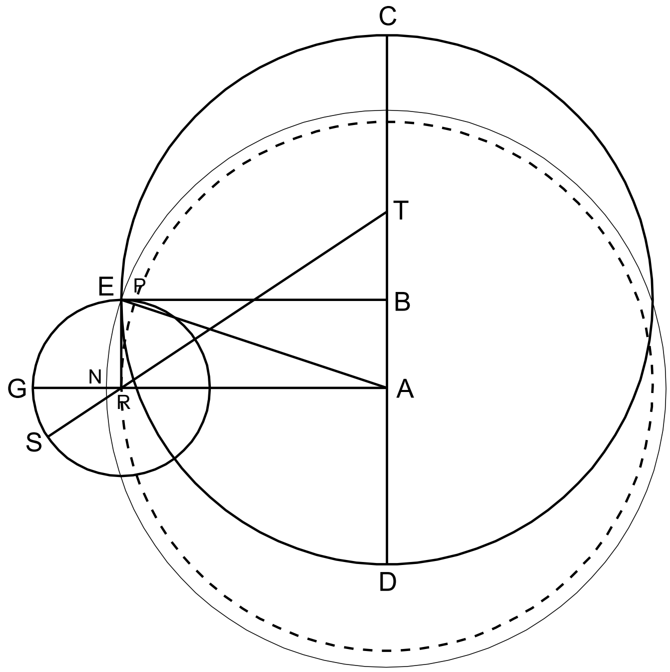
AN = AE = secant ∠AEB = secant ∠EAR

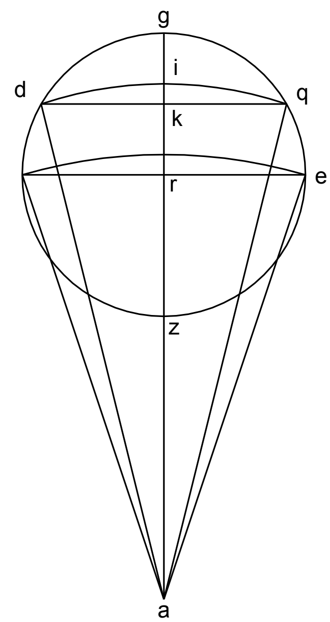
we should instead take

AR = EB = AP.

So instead of deciding on the right place for it to be by taking the secant, AE, the planetary mind instead considers that it should use the radius, EB, or AR, since that is what AP, the true solar distance it adopts, is equal to. Now this is a special case, where AR happens to equal the radius of the eccentric, EB, and it is also at right angles to the line of apsides.

Those conditions will not generally obtain for other points on the old orbit, and so it is not entirely clear *of what general rule the present construction is an instance*.

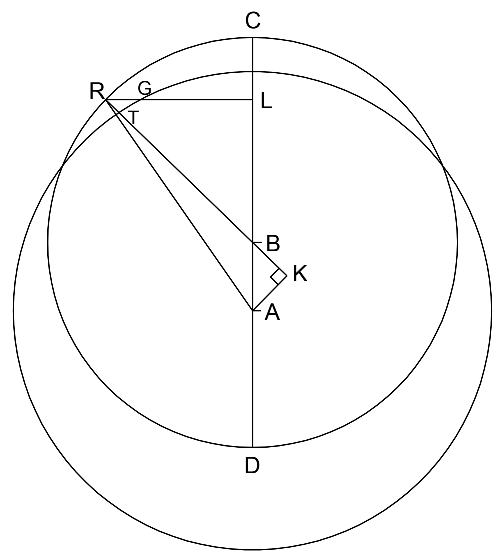




In this chapter, Kepler in fact gets the general rule WRONG! What he sees is this: P, the point which is truly on the orbit of Mars, is the intersection of AP and EB, and also AP is equal to EB. But away from this special place on the orbit, what are the more general descriptions of lines AP and EB that matter?

Kepler considers other points on the original theoretical orbit and asks what general rule of construction, beginning with those, can we infer from the special case we have discovered. Suppose R is a random point on the old theoretical circular orbit. Join RB, RA, and complete a right triangle on AR as hypotenuse by extending RB to K, where the line drawn from A at right angles to RB meets it. So now we have the following analogs:

Special case considered above	Case considered now
E	R
△EBA	△RKA
EA	RA
EB	RK
AP	?



Of what general rule was the construction of point P a special and simple instance? That is the question. Do we, for example, instead of taking AR as the right solar distance to Mars, take RK instead? That seems very plausible, given their analogy to EA and EB. Then do we make a circle around center A with radius RK? Again, that seems very plausible, since that is what we did with EB, which is analogous to RK. But now what? Suppose this circle we have drawn intersects the line RK at T, and the perpendicular RL (drawn to the line of apsides) at G. Which one of these points should be considered a point on the true orbit of Mars? Either one could make a claim to being analogous to point P in the special case we considered.

T is like P because it lies on the side of the right triangle, RK, which was analogous to EB, just as P lies on EB. So we should expect the true point of the orbit once again to lie on RK, the side of the right triangle which gives us the correct distance.

G is like P because it lies on the perpendicular from R to the line of apsides, just as P did in the special case we considered.

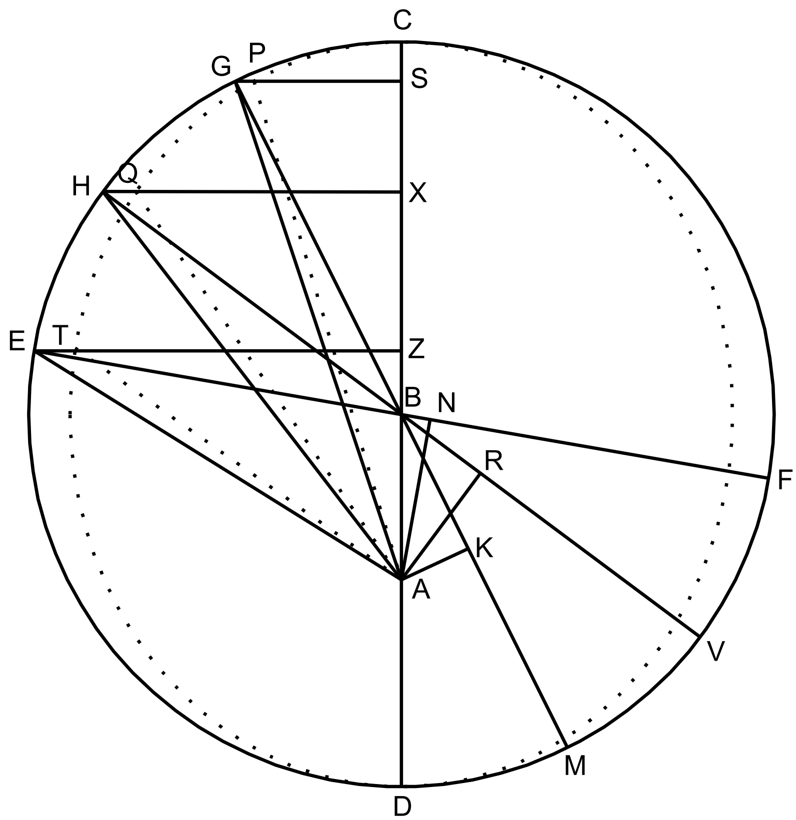
So which is it?

Here in Chapter 56, Kepler assumes that it is T. But in fact he is mistaken—it is G, and he corrects himself in Chapter 58.

And so that will give us a general rule for constructing points on the true orbit of Mars. For any point R on the circle about the line of apsides, we can construct a corresponding correct point on the true orbit of Mars thus:

- We drop RL at right angles to the line of apsides CD
- We join RB, RA, and complete the right triangle RKA, where RK passes through B.
- We draw a circle with center A, radius equal to KR.
- Where this circle intersects RL will be a point on the true orbit of Mars, as all the observations of both Brahe and Kepler confirm.

Here is a diagram illustrating other such points on the true orbit. The orbit itself is the dotted line, “squished in” from the circle, as it should be. The rule just given explains exactly how much the orbit should be squished. (The diagram does not represent the amount of squish precisely—we will present a scale diagram later to give a sense of just how “squished” the Martian orbit is.)



KEPLER

DAY 46

MAGNETISM THE MOST LIKELY CAUSE OF PLANETARY MOTION; HOW KEPLER DISCOVERED THE TRUE RULE FOR CONSTRUCTING POINTS ON THE ORBIT OF MARS

Chapters 57, 58

CHAPTER 57

BY WHAT NATURAL PRINCIPLES THE PLANET MAY BE MADE TO RECIPROCATE AS IF ON THE DIAMETER OF AN EPICYCLE

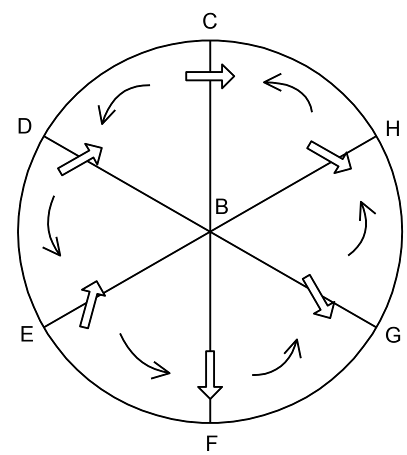
Here we are doing physics again. Kepler keeps speaking of the planet making some sort of reciprocation along the line joining it to the sun which is the equivalent of its being on an epicycle. But sometimes he is thinking just of the mathematical relationships, other times of physical causes that might produce these results, other times of the connection between the physics and the mathematics. His own mind reciprocates, one might say—that is the nature of the discovering mind, to look back and forth. That is the motion of searching, and also of uncertainty.

(1) THE RULE OF RECIPROCATION. Kepler begins the chapter by reintroducing the epicyclic diagram, suggesting that he will use it to ask questions about what the mind of the planet is doing, and after getting his physics straight. But he does not get into details about the rule for determining the reciprocation just yet. That will come up later, after his new physical ideas are brought forward. At the beginning of this chapter he says he will, by the end of Ch. 57, reject the idea of a planetary mind. "And since the finger points to a *natural* way of measuring this reciprocation, its cause will also be natural; that is, it will be some natural—or better, corporeal—faculty, and not a planetary mind." (William H. Donahue translation.)

QUESTION: What is the opposition between the manner in which Mind operates, and that in which Nature operates, which Kepler is implying here? Nature acts uniformly, one definite fixed way, in accord with rules of intensity determined by proximity to the causal body; also it cannot do otherwise than it does, but does as much as it can, as soon as it can, and as long as it can; it does not adjust its power or reaction to some special purpose. And it is easy and smooth. Kepler seems to be thinking of a mind that has to perform calculations, like our own, which would therefore be clumsy, and hesitant.

Here he also enunciates this physical PRINCIPLE: The planet by itself cannot move from place to place unless assisted or directed by an extrinsic force (since it has no feet, wings, etc.). That was from Ch. 39. Therefore "we must as a consequence also ascribe this reciprocation in part to the solar power." So now he will introduce a way to understand how the Sun can in some way cause the "reciprocation."

(2) THE CIRCULAR RIVER. Kepler superimposes a diagram of a circular river upon our orbit (which is traced out by the star on the epicycle). He imagines a boat with an oar in the water, being oriented in different ways. We might also imagine a sailboat in a circular windstorm on the water.



The sailor rotates his sails (or oar, or rudder) 180° for every time the wind (or water) carries the boat 360°. So "the sailor revolves his oar once in twice the periodic time of the" ship which is analogous to the planet.

Imagine our sailor beginning at F, with his sails perpendicular to the wind. Then he is moving fastest there, being most effectively acted on by the wind. But as he goes to G and H, he rotates his sail counterclockwise to that by the time he is at C he has moved it 90°. His sail is now parallel to the wind, so that the boat moves very slowly.

Moving on toward intermediate points D and E, he continues to rotate the sail in the same direction, so that the boat is speeding up again (since the sails are becoming more perpendicular to the movement of the water). By the time he gets back down to F, his sail will have turned 180°.

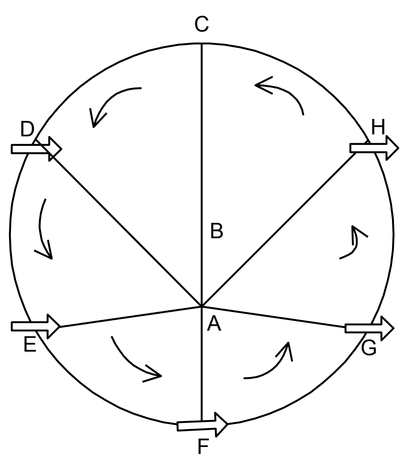
(3) SHORTCOMING IN THE CIRCULAR RIVER EXAMPLE. We had to make the oar (or sail) rotate at half the speed of the boat around the river, so that the oar (or sail) would always be perpendicular to the current at F, and parallel to it at C (if it turned 180° after 180° of travel in the river, it would be perpendicular in both places). But we want the planet to have the same speed on the epicycle as the epicycle does on the deferent!

Also, Kepler observes, while the force of a river is material, the force of the sun is "immaterial." It is not altogether clear what Kepler means by this. The influence of the sun is invisible, but then again, so is air, and it is not "immaterial." Probably he is thinking that since the periods of the planets increase as one goes out from the sun, therefore the lines of influence whirling about with the rotation of the sun, presumably with the same period as the sun's rotation, have less efficacy further from the sun, yet they still affect the planets and cause them to orbit. It is as if the sun's immaterial species were lines rotating with it, but they moved right through the planetary bodies, gently paddling them along. The further away from the sun, the more diffuse or attenuated these immaterial species become, and the less quickly they move the planet.

(4) A BETTER EXAMPLE: MAGNETS. Kepler says that from this very refutation comes another example which will work better. The river and the oar are alike in both being material, so since the Sun's power is immaterial, let's make the oar of the planet immaterial, too. What if all the planets are enormous round magnets? Of Earth there is no doubt. William Gilbert has proved it.

So Kepler posits a magnetic axis in the planet, giving it a North (indicated by the arrow-head in the diagram reproduced below) which seeks the Sun, and a South (the arrow-tail) which flees the Sun. And the axis just stays parallel to itself. This is a bit physically troubling, but he finds support for it by a precedent: Its daily-rotation axis stays (more or less) parallel to itself all the time as it orbits the Sun. Here Kepler finds fault with **Copernicus** for thinking he needed a special principle to cause the Earth to rotate its axis at the same speed and in the opposite direction of its angular movement around the Sun. (This is the "piece-of-gum-on-a-record" image I use to explain Copernicus's idea back in Day 32.) Kepler thinks this is superfluous, and it is better to say that Earth's axis stays parallel to itself because that is natural, or what it does when no extrinsic causes are introduced to shift it. That notion is much closer to the modern idea of inertia.

(5) THE MOVEMENT OF THE APHELIA. Kepler says he has one more motion left to explain by his magnetic hypothesis, namely the "extremely slow progression of the aphelia," i.e. the precession of the planetary orbits (in our case, this is the same as the precession of the equinoxes). But we have no need to trouble ourselves with that detail.



CHAPTER 58

IN WHAT MANNER THE RECIPROCATION DISCOVERED AND DEMONSTRATED IN CHAPTER 56 MAY BE ACCEPTED, AND NEVERTHELESS AN ERROR MAY BE INTRODUCED IN A WRONGHEADED APPLICATION OF THE RECIPROCATION, WHEREBY THE PATH OF THE PLANET IS MADE PUFF-CHEEKED

(1) NATURE PLAYS HARD TO GET. Kepler opens this chapter with the following line: "Galatea seeks me mischievously, the lusty wench. She flees to the willows, but hopes I'll see her first."

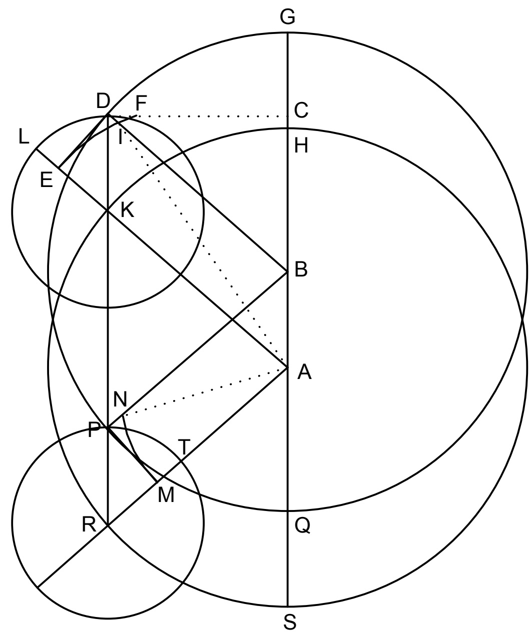
What an opening!

He is quoting Vergil (from the *Eclogues*, 3.64). Kepler himself explains his reason for quoting this line: "It is perfectly fitting that I borrow Vergil's voice to sing this about Nature. For the closer I approach to her, the more petulant her games become, and the more she again and again sneaks out of the seeker's grasp just when he is about to seize her through some circuitous route. Nevertheless, she never ceases to invite me to seize her, as though delighting in my mistakes." (William H. Donahue translation.)

The pre-Socratic philosopher said that "Nature loves to hide."

Kepler more colorfully says "Nature *plays hard to get.*"

(2) THE AIM OF THE WHOLE BOOK is stated with remarkable clarity and brevity in the second paragraph of Ch. 58. "Throughout this entire work, my aim has been to find a physical hypothesis that not only will produce distances in agreement with those observed, but also, and at the same time, sound equations," he means ANGLES, "which hitherto we have been driven to borrow from the vicarious hypothesis of Ch. 16." He is repeating what he said in the introduction to the book. The goal is to find really-existing and plausibly-operating physical causes for the motions of the stars which will produce the correct *distances* and the correct *angles* for the location of a planet at given times. The coherence of the physics with the mathematics, and the necessity with which the physics produces the mathematics, he takes to be a sign of the truth. The true can follow from the false, but not in that way!



(3) HOW KEPLER DISCOVERED THE TRUE RULE FOR CONSTRUCTING POINTS ON THE TRUE ORBIT OF MARS.

Kepler draws the diagram again for his epicyclic hypothesis.

A is the Sun.

QG is the Martian line of apsides.

B is the midpoint of QG, and the geometric center of the perfect circle GDQ, our good old circular hypothesis.

C is the equant-point.

HKRS is another, equal circle, center A (the Sun), which is the deferent in the epicyclic equivalent to our good old circular hypothesis. K and R are two different positions of the epicyclic center, with the epicyclic radius DK = RP = AB (the eccentricity), and DK always remains parallel to itself, so that it describes the circle of the hypothetical orbit.

Now drop DE perpendicular to EA, producing the specific "reciprocation" LE.

Kepler says he had already become convinced, by Ch. 56, that AE is a true distance to the Martian orbit—but where does that distance occur? Or rather, WHEN does it occur? In Ch. 56, he thought the rule was to take AE as a radius, A as center, and draw a circle cutting AD at I, so that AI = AE, and point *I* would be a point on the true orbit of Mars.

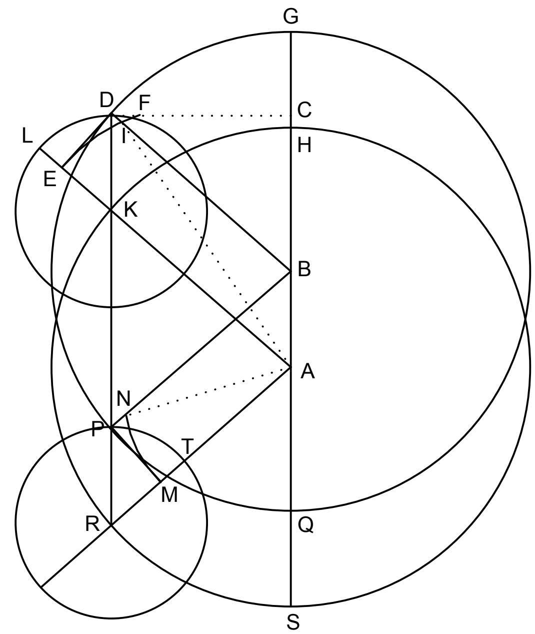
Here he shows that this was a mistake. The same circle will cut the perpendicular DC a little higher up, at F, and *that* turns out to be a point on the true orbit of Mars. As we noted in Day 45, he originally picked the wrong rule.

How does he determine this? By the TIMES each rule determines, since each determines AREAS around the Sun.

If the planet were at *I* after the same time it takes the epicycle (in the version which is equivalent to the old circular orbit) to bring the planet to D, then by the area-law, this time is proportional to the area DAG in our simple circular orbit. If we now take that amount of time (i.e. that same portion of the Martian period), that tells us exactly WHEN Mars should be at *I.* This is what Kepler means by saying he "fitted the angle IAG, rather than FAG, to this area DAG, now converted to time," i.e. he paired the angle IAG with the time proportional to DAG. That done, he found that the angle IAG disagreed with those determined by reasoning from observations, i.e. that angle was not the one Mars actually had at the appointed time.

This is how he discovered that I was the wrong point, and F was the right one.

(4) THE FINAL MISTAKE: THE "PUFF-CHEEKED" ORBIT BASED ON THE SLIGHTLY ERRONEOUS RULE OF CH. 56.



Kepler shows that the orbit produced by taking the wrong rule, the rule of point I, is "puff-cheeked." I think we are to imagine a pear-shape. What he shows, geometrically, is that if we assume the "I-rule" is correct, and if we take the epicycle at K and R so that arcHK = arcRS, and hence also arcGD = arcQP (or ∠GBD = ∠QBP), and if we draw the points on the orbit (according to the "I-rule") for the locations of the epicycle at K and R (let these be I and N), and if we draw BID through to the simple circular hypothesis, and BNP through, then, even though ∠GBD = ∠QBP, nevertheless PN > DI.

This asymmetry would not happen if the path were an ellipse. So had Kepler known the path was an ellipse, he would very easily have realized that the "I-Rule" is wrong, since it produces an asymmetrical path, and therefore not an ellipse.

He proves that PN > DI as follows.

∠DKE = ∠BAK (DK parallel to AB)
∠PRM = ∠PBA (opp. angles in parallelogram PRAB)
but ∠PBA = ∠BAK (arcGD = arcQP)
so ∠DKE = ∠PRM
but KD = PR (both are equal to AB, the eccentricity)
and ∠DEK = ∠PMR (both are 90°)
so △DEK is congruent to △PMR
so ED = MP.

But ∠EDI = ∠MPN (both are 90°, since ED is perpendicular to BD, MP to BP)

And AE > AM (since AK = AR, but AE is greater than these, AM less)

so circle radius AE (or AI) is greater than circle radius AM (or AN)

Now consider what we have.
AMP is a right angle, and so is AED.
MP = ED.
but AM < AE.
So what happens when we draw the circles on radii AM, AE, cutting the parallels DB and PB at *I* and N?

PN > DI.

This could be proven, but Kepler takes it as obvious.

So the orbit is "puff-cheeked," i.e. fatter down in its lower region than it is in the symmetrical point in the upper region.

(5) KEPLER DID NOT RECOGNIZE THE ELLIPSE.

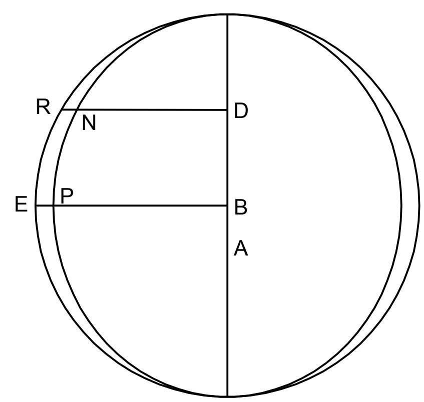
Kepler, understandably, did not at first recognize the ellipse as the LOCUS of points produced by the new, correct rule (where we take the intersection of the circles with the perpendiculars). He is harder on himself, though: "O ridiculous me! To think that the reciprocation on the diameter could not be the way to the ellipse!"

He says he already knew that the ellipse produced the right equations, but he did not recognize the ellipse as the locus of the points produced by the rule of reciprocation. But even at that point he did seem to know that the true orbit was symmetrical top-and-bottom, since he rejects the "puff-cheeked" orbit, resulting from the incorrect rule for finding the angle of the correct distances, on the grounds that it is not symmetrical top-and-bottom.

(6) THE THOUGHT UNDERLYING THE EPICYCLIC EQUIVALENT.

It might seem amazing to us that Kepler keeps bringing us back to an epicycle, when he himself does not believe in the physical possibility of an epicycle. And we might think that he was slow to have done with epicycles simply because he was coming from a world in which epicycles abounded in astronomy. But his real preoccupation with it has to do with his physics. The true orbit of Mars is so extremely close to being a circle that it seems natural and inevitable to consider it as an "adjusted circle." But what could possibly be the cause of this "adjusting"?

Since the orbit of Mars will turn out to be a nearly-circular ellipse (as we shall see), one could easily define its orbit as an "adjustment" of the circle whose diameter is the line of apsides. That is, if B is the center of our original circular hypothesis, BE the radius at right angles to the line of apsides, P the point where the true (elliptical) orbit of Mars cuts it, it is true to say also that if we choose any other random point R along the circumference of the circle and drop RND at right angles to the lines of apsides, then



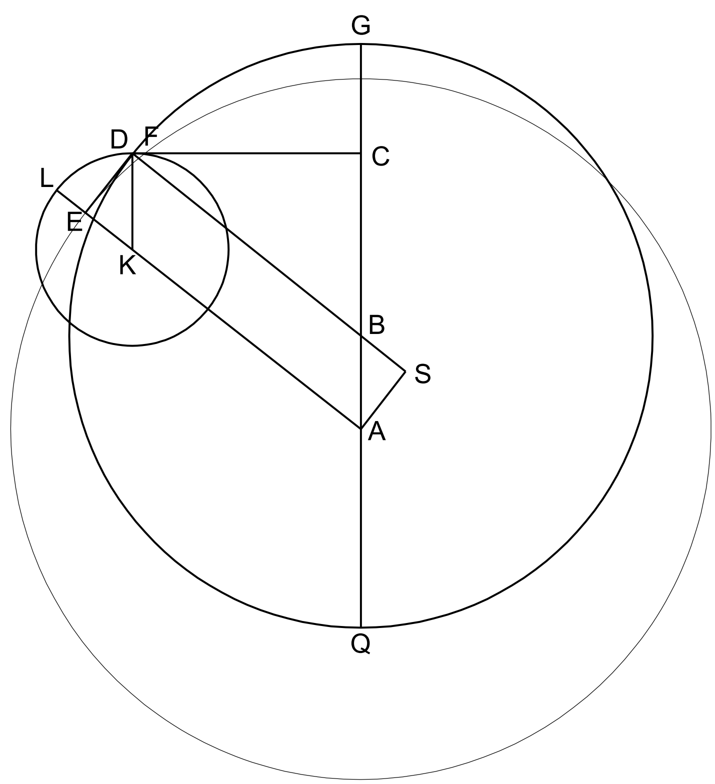
RD : ND = EB : PB

We will prove this later. But for now it is enough to see that there is something quite true about calling the orbit of Mars an "adjusted circle." And yet the kind of adjustment just defined hardly illuminates the physical cause of it!

The only candidate causes are Mars and the Sun, the two corporeal entities in question possessed of causal powers. So let the location of hypothetical Mars, on the old circle, be D, and join DB. That is, as it were, Mars's default position until we introduce an "adjustment." How do Mars and the Sun cooperate so as to produce F, the corresponding point on the true orbit of Mars? Well, mathematically, if we extend DB to S where AS is perpendicular, the length DS will be equal to AF.

And if we join AK and extend it to E where DE is perpendicular to AE, the length AE will be equal to AF, too. That is the epicyclic's equivalent way of producing the right distance.

Kepler seems to be haunted by the fact that the length of AF is thus the product of ghosts. After all, there is nothing at K or D or E or B or S. But at least there is something at A—the Sun! Kepler is at pains to explain how length AE or SD (and then the use of it as a radius, intersecting the perpendicular DC, or, in the erroneous rule of Ch. 56, intersecting the line BD) can result from MAGNETIC interaction between Mars and the Sun. Since Kepler's magnetic theory is finally not correct, we will not go into further detail on it. A more correct understanding of the physics of celestial motions must wait until Newton. It is really only with his help that we become fully emancipated from the circle, in which the elliptical orbit of Mars is understood in its own right as the product of inertial velocity and gravitational pull, and not as an "adjusted circle."



KEPLER

DAY 47

BASIC GEOMETRY OF THE ELLIPSE IN PREPARATION FOR DEMONSTRATING THAT THE ORBIT OF MARS IS AN ELLIPSE

Chapter 59

Chapter 59 is the big one—the chapter in which Kepler will demonstrate that the orbit of Mars is in fact an ellipse. He does not in this chapter mention that the Sun is at the focus, although he is the one who realized this, and in fact he gave that particular point in the ellipse the name "focus." More on this later.

Ideally, a reader of the *Astronomia Nova* would already be familiar with the geometry of conic sections as presented in the first three books of *On Conic Sections* by Apollonius of Perga. But since that book is not very widely read, I will try to supply what knowledge of conics Kepler assumes on our part. Actually, the argument that the orbit of Mars is an ellipse requires only one very simple theorem which we can prove directly from a cone, without all the preceding theorems in Apollonius. In proceeding this way, I will be imitating Galileo, who did much the same thing in his *Two New Sciences*, when proving certain fundamental things about parabolas to readers unfamiliar with Apollonius.

I will also supply a quick and elegant derivation of the chief focal property of the ellipse, again from the cone, and without the mass of preliminary propositions which Apollonius requires for the same. In each case what makes the great simplicity possible is the fact that I will be using a right cone, and talking only about the axis of an ellipse, whereas Apollonius is interested in making his theorems apply to all cones, oblique ones included, and he is also keen to show that many of the properties of the axis belong also in some way to every diameter in the ellipse. But we are not studying the ellipse for its own sake, here, and so we can afford to give up generality for the sake of brevity and clarity.

Here in Day 47, then, I will first show four things:

LEMMA 1. Whenever you cut a right cone with a plane, the section produced has an axis, that is, a straight line within it which bisects all the straight lines drawn at right angles to it and from one side of the section to the other.

(These straight lines drawn at right angles to the axis and bisected by it are said to be "**drawn ordinatewise**" to the axis, and their halves are called "**ordinates.**")

LEMMA 2. Whenever you cut a right cone with a plane so as to produce a closed section, the squares on any two ordinates to the axis are to each other as the rectangles contained by the segments into which those ordinates divide the axis.

LEMMA 3. In a right cone, if a closed section is formed by a cutting plane which is not parallel to the base of the cone, i.e. not at right angles to the axis of the cone, then the closed section is not a circle.

(These non-circular closed conic sections are called "**ellipses**.")

LEMMA 4. If in a right cone containing an ellipse two spheres be inscribed, each one tangent to the cutting plane at a point and to the cone's surface at a circle, one above and the other below the cutting plane, then the two points of tangency lie on the major (greater) axis of the ellipse, and the sum of the straight lines drawn from them to any point on the ellipse is equal to the axis of the ellipse.

(These two points are called the "**foci**" of the ellipse; each one is called a "**focus**" of the ellipse.)

Now let's prove these things . . .

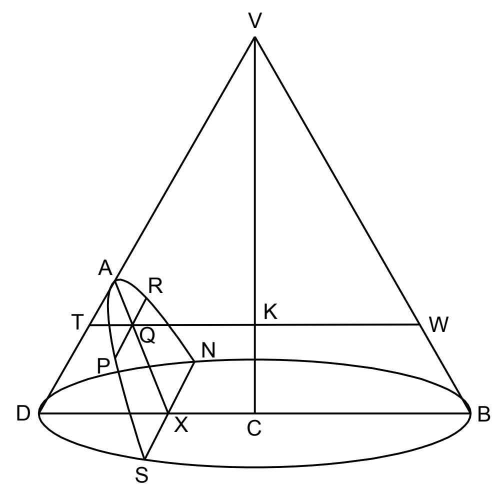
LEMMA 1. Whenever you cut a right cone with a plane, the section produced has an axis, that is, a straight line within it which bisects all the straight lines drawn at right angles to it and from one side of the section to the other.

(These straight lines drawn at right angles to the axis and bisected by it are said to be "**drawn ordinatewise**" to the axis, and their halves are called "**ordinates**.")

Let V be the vertex of a right cone (i.e. a cone whose axis VC is at right angles to its base circle, center C).

If the cutting plane is parallel to the base circle, then the common section of it and the conic surface will clearly be another circle, in which case the conic section obviously has an axis, since every diameter of a circle is an axis—that is, a straight line in the circle which bisects all the straight lines drawn at right angles to it and from one side of the circle to the other.

If the cutting plane is not parallel to the base circle, then it will cut the base circle (or a circle parallel to it) somewhere inside the cone, say along the straight line SN. And the common section of the cutting plane and the conic surface will be a continuous line such as SAN. I say that this conic section, too, has an axis.



Draw CX at right angles to SN in the base circle.

Extend XC to D and B at opposite ends of the base circle; hence DXCB is a diameter of that circle.

Clearly if we join VD and VB (which straight lines lie on the conic surface), since they intersect at V, it is not possible for the cutting plane to be parallel to both, and hence it cuts at least one of them—say VD, at A.

Join AX (which lies in the plane of △VBD).

I say that AX is an axis of the conic section.

Choose any point P on the conic section, and draw PQR at right angles to AX, that is, *parallel to* SXN.

Through Q, and in the plane of △VBD, draw TQW parallel to DXB (which therefore cuts VC, say at K).

Therefore the plane through PQR and TQW is parallel to the plane through SXN and DXB. But the planes parallel to the plane of SXN and DXB, i.e. to the plane of the base circle, obviously make circles on the conic surface. Therefore T, R, W, P lie on a circle.

But just as DCB is the diameter of the base circle, and C is its center, so too TKW is the diameter of the circle through T, R, W, P, and K is its center.

Now, since	SXN is at right angles to DXB	[construction]
and	PQR is parallel to SXN	[construction]
and	TQW is parallel to DXB	[construction]
thus	PQR is at right angles to TQW	

But PR is the chord of a circle and TQW is its diameter.

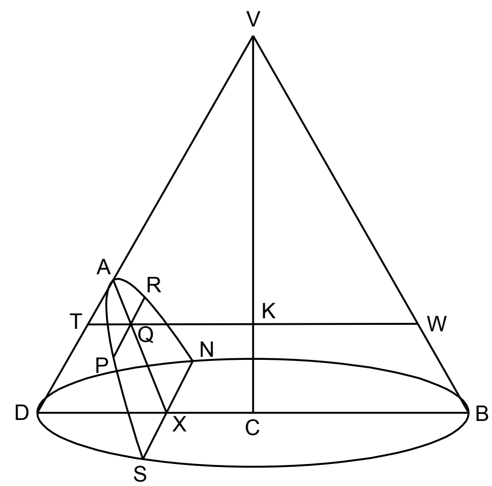
Thus PR is bisected at Q.

Therefore, taking a random point P on the conic section, and drawing a perpendicular from it to our line AX and through to the other side of the section, AX bisects it.

Therefore AX is an axis.

Q.E.D.

Vocabulary:



AX is called an "**axis**" of the conic section, that is, a straight line passing through the section which bisects all the chords to the section which are at right angles to itself.

Lines such as PQR and SXN, drawn at right angles to the axis and bisected by it, are said to be "**drawn ordinatewise**" to the axis, and each of their halves, for instance PQ or SX, is called an "**ordinate**."

And any triangle such as VBD, with the vertex of the cone as one of its vertices, and having for its other two vertices the ends of a diameter of the base circle of a right cone, is called an "**axial triangle**" in the cone, since such a triangle passes through the axis of the cone VC.

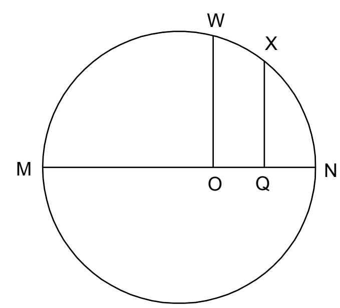
PORISM: Hence it is clear that *in a right cone, the axis (AX) of a conic section (SAN) is the intersection of the cutting plane (SAN) with that axial triangle (VBD) which cuts the intersection of the cutting plane and the base plane (SN) at right angles (i.e. BCD is at right angles to SN)*.

LEMMA 2. Whenever you cut a right cone with a plane so as to produce a closed section, the squares on any two ordinates to the axis are to each other as the rectangles contained by the segments into which those ordinates divide the axis.

Not every conic section is a closed figure—parabolas and hyperbolas open up forever. But we are not concerned with them for the moment. We are concerned with the ones which form closed figures.

That happens whenever the cutting plane cuts both sides of an axial triangle VBA—for example, the circle CK (in cone below) cuts both sides of the axial triangle, namely VB and VA, and so does the figure ES.

In closed-figure conic sections like these, the squares on the ordinates are to each other as the rectangles contained by the segments of the axis into which they respectively divide the axis.

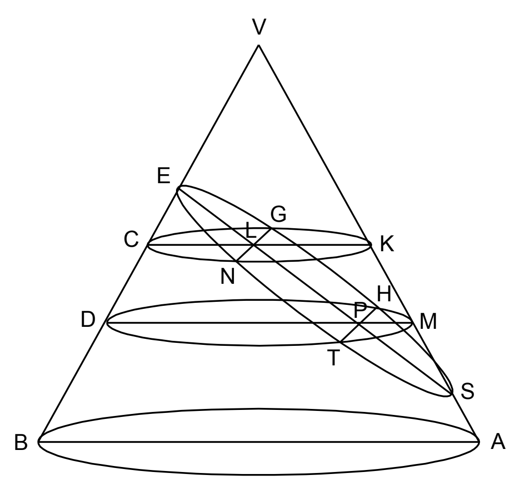


If the section is parallel to the base circle, then it is itself a circle, and then this theorem is obvious, since in the case of a circle, the square on any ordinate is equal to the rectangle contained by the segments into which it divides the axis (or diameter). For example, if MN is a diameter of a circle, and OW and QX are drawn ordinatewise, then

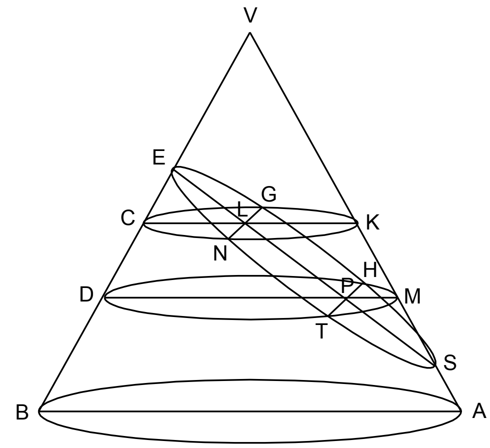
$OW^2 = MO \cdot ON$
and $QX^2 = MQ \cdot QN$
so $OW^2 : QX^2 = MO \cdot ON : MQ \cdot QN$

If the section is not parallel to the base circle, then it will be cut by planes drawn parallel to the base circle.

Let V be the axis of a right cone, base circle of diameter AB. Let ES be a closed conic section for which ES is the axis, and VAB the axial triangle in whose plane ES lies (Lemma 1), and let it be that ES is not parallel to the base circle. Then I say that the squares on the ordinates are still as the rectangles contained by the segments into which the axis is divided by the ordinates. For take any two points on the section, N and T, and through these pass planes parallel to the base plane. Therefore these planes will be circles intersecting the cutting plane in straight lines as GN and HT, which are ordinates to the axis (Lemma 1), and which are thus parallel to each other and also bisected by ES, as at L and P. But these ordinates also lie in the planes of the parallel circles, and are with equal reason ordinates in the circles to the diameters CK and DM.



Now (EL : EP) c (LS : PS) = (EL : EP) c (LS : PS)
[obviously]
so (CL : DP) c (LK : PM) = (EL : EP) c (LS : PS)
[similar triangles ELC and EPD, KLS and MPS]
so CL·LK : DP·PM = EL·LS : EP·PS
[forming the rectangles]
i.e. $NL^2 : TP^2$ = EL·LS : EP·PS
[property of the circles]



Q.E.D.

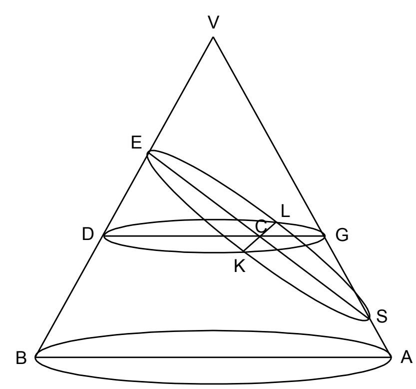
NOTE: This theorem corresponds to Apollonius's Theorem 21 of Book 1 of his *Conics*.

LEMMA 3. In a right cone, if a closed section is formed by a cutting plane which is not parallel to the base of the cone, i.e. not at right angles to the axis of the cone, then the closed section is not a circle.

(These non-circular closed conic sections are called "**ellipses**.")

Let V be the vertex of a right cone, ES a closed section not parallel to the base circle, ES its axis, △VBA the axial triangle in which ES lies.

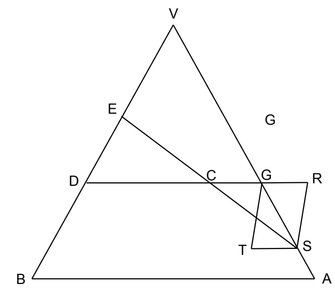
I say that section ES is not a circle.



For let ES be bisected at C, and through C pass a plane parallel to the base circle, thus producing circle DKGL with KCL a common ordinate to both the conic section in question and the circle whose diameter is DG (which is parallel to AB). Clearly, if section ES is a circle, then C would have to be its center, since ES is its axis. And thus KC would have to be equal to EC and CS, and hence KC^2 would have to be equal to EC·CS. Hence proving that KC^2 < EC·CS would be sufficient to prove that section ES is not a circle.

Now ∠BDC < ∠DEC [exterior angle of △DEC]
so ∠AGC < ∠DEC [∠AGC = ∠BDC since the cone is right]

i.e. ∠SGC < ∠DEC



So if we draw ∠CGT = ∠DEC, then GT falls inside the cone, and hence if we draw SR parallel to GT, then SR falls outside. So now

∠CRS = ∠CED
thus △DEC is similar to △CRS
so DC : CS = EC : CR
so DC·CR = EC·CS

so DC·CG < EC·CS [since CG < CR]
so KC^2 < EC·CS [since KC^2 = DC·CG by the circle]

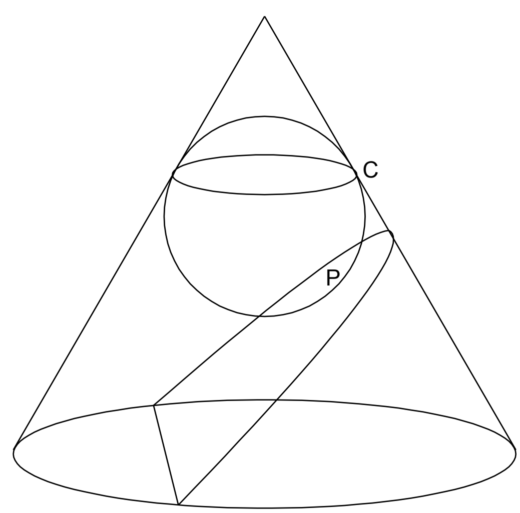
Hence section ES is not a circle.

Q.E.D.

Such a section is called an "**ellipse**."
And, since KCL is another axis of the ellipse, but it is less than ES, it is called the "**minor axis**," i.e. the lesser one, and the original axis we discovered, ES, which lies in the plane of an axial triangle, is called the "**major axis**," i.e. the greater one.
LEMMA 4. If in a right cone containing an ellipse two spheres be inscribed, each one tangent to the cutting plane at a point and to the cone's surface at a circle, one above and the other below the cutting plane, then the two points of tangency lie on the major (greater) axis of the ellipse, and the sum of the straight lines drawn from them to any point on the ellipse is equal to the axis of the ellipse.

(These two points are called the "**foci**" of the ellipse; each one is called a "**focus**" of the ellipse.)

The focal properties of conics are beautifully and memorably derived by means of the "Dandelin Spheres" (named for the Belgian mathematician Germinal Pierre Dandelin) in a right cone. What is a "Dandelin Sphere"? If we have a right cone, a sphere "dropped in" will be tangent to the conic surface at a circle, C, which is parallel to the base circle. And if such a sphere be tangent to the cutting plane producing our conic section, it will be tangent to it at a single point P. Such a sphere is called a "Dandelin Sphere," and by means of it we can derive the main focus property of the ellipse (in fact, all the basic focal properties of all the conics, but we will not go that far).



The Dandelin-Sphere derivation of the focus property of an ellipse.

Given an ellipse with major axis AB, let it be included in a right cone of vertex V, with the plane of the axial triangle VMK producing axis AB.

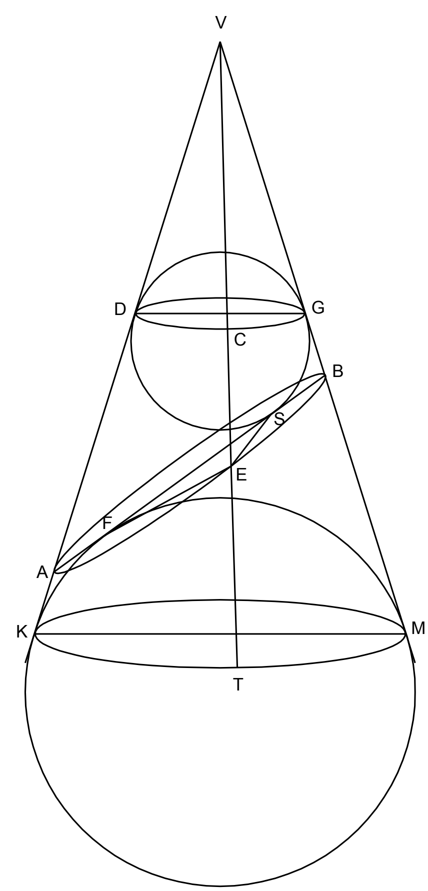
Let Dandelin Spheres be inscribed, one tangent to the conic surface above the ellipse at circle DG and to the plane of the ellipse at S, another tangent to the conic surface below the ellipse at circle KM and to the plane of the ellipse at F.

Choose a point E at random on the ellipse.

I say that FE + ES = AB

Join VE, cutting circle DG at C and circle KM at T.

Now	SB = BG	[being 2 tangents to 1 sphere from 1 point]
and	FB = BM	[again, tangents]
so	SB + FB = GM	
And	FA = AK	[again, tangents]
and	SA = AD	[again, tangents]
so	FA + SA = KD	
But	GM = KD	[given the parallel circles and the right cone]
so	SB + FB = FA + SA	
so	SB + (FS + SB) = FA + (FS + FA)	
so	2SB = 2FA	
so	SB = FA	[thus F, S are equidistant from A, B]
But	GM = SB + FB	[first result above]
so	GM = FA + FB	
so	GM = AB	
or	CT = AB	
Now	EF = ET	[again, tangents]
and	ES = EC	[again, tangents]
so	FE + ES = CT	



So F and S have the familiar focal property you might have learned in high school.

Q.E.D.

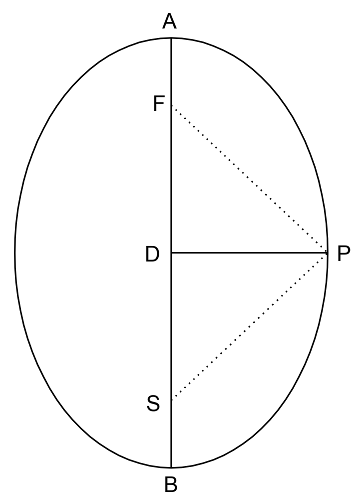
COROLLARY 1: The foci are equidistant from the midpoint of the major axis (or from the ends of the major axis). That is, if D is the midpoint of the major axis,

FD = DS

For it was shown in the course of the proof that

SB = FA

COROLLARY 2: If AB is the major axis of an ellipse, and D is the midpoint of the axis (the "center" of the ellipse), and DP is set up at right angles to the major, and hence is the half the minor axis (i.e. the "semi-minor"), and if F and S are the foci, then FP + PS = AB, by the focal property. But since F and S are also equidistant from D (Cor. 1), the midpoint of the axis, thus FP = PS. Hence each of these lengths is equal to half the major. Hence if P is the endpoint of the semi-minor, and PF = semi-major, then F must be the focus.



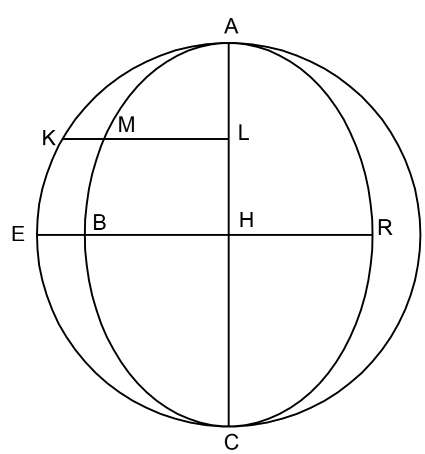
KEPLER

DAY 48

Chapter 59
Protheorems I – VII

Kepler himself develops some geometry of the ellipse as a preliminary to his proof that the orbit of Mars is an ellipse. These preliminaries he calls by the odd name of "protheorems." Perhaps what he means by the word is as follows: just as a "proposition" is a "position" which is laid down for the sake of proving a further position, that is, it is a premise, so too a "protheorem" is a "theorem" which is put forth for the sake of generating a further theorem from it. So a "protheorem" is like a lemma, except that unlike most lemmas, a protheorem is to some extent interesting in itself. The word "lemma," recall, is derived from a Greek verb meaning "to take (for granted)." He proposes twelve of these protheorems, but for today we will only look through the first seven. Even this will be going a bit overboard, since we only need Protheorems 1, 6, and 7 in order to follow the argument showing that the orbit of Mars is an ellipse.

PROTHEOREM 1



Given: Circle H, diameter AC
Ellipse H, major AC, minor BR
Random perpendicular KML

Prove: LM : LK = BR : AC

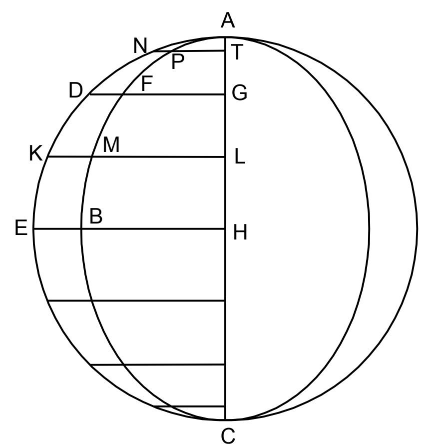
$LM^2 : BH^2 = AL \cdot LC : AH \cdot HC$
[Lemma 2, Day 47]
$LK^2 : EH^2 = AL \cdot LC : AH \cdot HC$
[circle property]
so $LM^2 : BH^2 = LK^2 : EH^2$
so LM : LK = BH : EH
but BH : EH = BR : AC [halves are as doubles]
so LM : LK = BR : AC

i.e. any such lines as LM, LK cut by the circumference of the ellipse and of the circle are to each other all in the same ratio, i.e. as the minor to the major.

Q.E.D.

NOTE: The converse is also true, i.e. that if you have a locus of all the points that cut the ordinates to an ellipse (or circle) in the same ratio, that locus is an ellipse (or a circle, if you get lucky).

PROTHEOREM 2



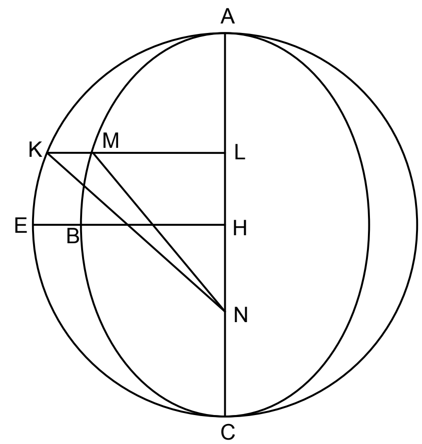
An ellipse is to its auxiliary circle (i.e. the one whose diameter is the major axis) as the minor is to the major.

Kepler refers to Archimedes's treatise *On Spheroids*, but the argument there is prolix and tedious. The Newton-style method is much easier: Cut the area of both the circle and the ellipse into as many tiny pieces as you like with thousands of ordinates, and then draw inscribed rectangles on them. Since any one rectangle in the semicircle will be under the same height as the corresponding rectangle in the semi-ellipse, these will be as their bases, e.g. as KL : LM. But that ratio is the same for all, i.e. it is the same as the ratio EH : BH, or as major to minor, as proved in Protheorem 1. *Componendo*, the sum of all the rectangles in the semicircle is to the sum of all in the semi-ellipse also in that ratio. But these rectangle-sums, always as the major to minor, each get as close as we please to the area of the semicircle and semi-ellipse. Therefore those two areas are also as major and minor. (If two areas are always as 5 : 7, no matter how close they get to being equal to A and B respectively, then the ratio of A : B must be the same as 5 : 7.)

PORISM: This same argument works for any two areas which are cut off by an ordinate in the two figures, e.g. Area AKL : Area AML = Major : Minor.

Q.E.D.

PROTHEOREM 3



NOTE: This is not needed for the argument for the elliptical orbit, but it is needed for Protheorem XV.

Given:	Circle H, diameter AC Ellipse H, major AC Random perpendicular KML Random N on AC
Prove:	Area AMN : Area AKN = Minor : Major

Well,	Area AML : Area AKL = ML : KL	[Protheorem 2, Porism]
but	$\triangle$NML : $\triangle$NKL = ML : KL	[Euc. 6.1]
so	Area AMN : Area AKN = ML : KL	[adding terms in first ratios]
but	ML : KL = Major : Minor	[Protheorem 1]
so	Area AMN : Area AKN = Major : Minor	

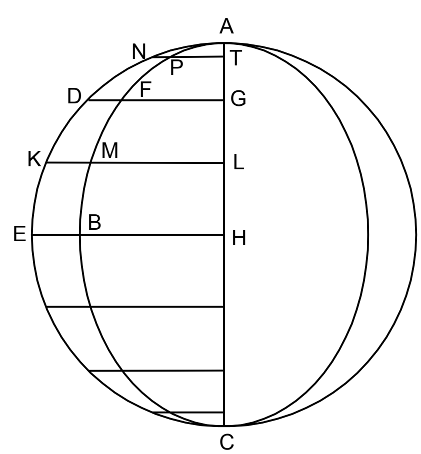
Q.E.D.

PROTHEOREM 4

Given:	Circle H, diameter AC arc AN = arc ND = arc DK = arc KE (any # of equal arcs) NPT, DFG, KML, EBH each perpendicular to AC
Prove:	arc BM and arc EK are more equal than arc NA and arc PA

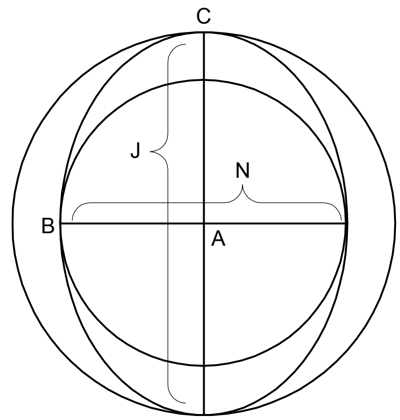
For, close to vertex A,

arc NA : arc PA = NT : PT (very nearly)



So the elliptical arc PA becomes less than the circular arc NA more and more in the ratio of PT : NT, as we take AT smaller and smaller. But as we get down to the middle arcs, like KE and MB, the arcs are nearly equal. "This is self-evident," says Kepler. Good enough, let's say.

PROTHEOREM 5



The entire elliptical circumference is approximately the arithmetic mean between the circle on the greater diameter and the circle on the smaller diameter.

So draw a circle on the major axis as diameter, and another on the minor axis as diameter, and Kepler is saying: The circumference of the ellipse is close to being the arithmetic mean of the two circular circumferences (i.e. half their sum).

He cites Ch. 48 (which we don't read) and also Archimedes' *On Spheroids* Prop. 7.

(a) Archimedes showed (he says) that the area of a circle drawn on the mean proportional between the 2 axes of an ellipse is equal to the area of the ellipse.

[That is not too difficult. The ellipse is to the circle on its major as minor to major (from Protheorem 2); but circles are as squares on their diameters; so what circle will be to the circle on the major as the minor to the major? Not the one on the minor, since that will be to the one on the major as the *square* on the minor to the *square* on the major. So we have to find a circle the square of whose diameter is to the square of the major as minor to major. Call the major J, the minor N. Obviously if we construct √JN, the square of this will have to the square of J the ratio JN : JJ, i.e. N : J, i.e. of minor to major. So the circles on √JN and J will be as minor to major. But the ellipse is to the circle on J as minor to major. Hence the circle on √JN is equal to the ellipse.]

(b) Kepler showed (he says, referring to Ch. 48) that the circumference of the ellipse is longer than that of this circle.

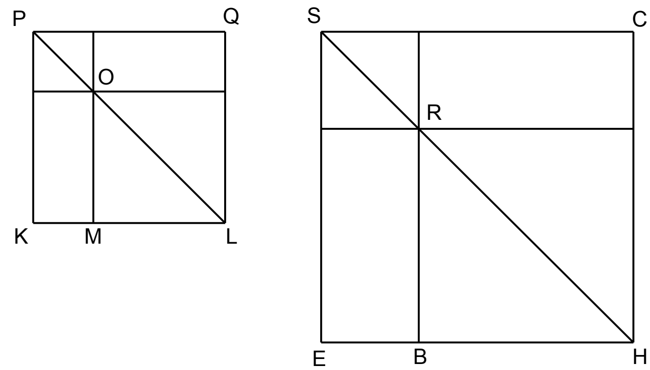
[This is also not too difficult. The circle and the ellipse have the same area. But the circle is more uniform, and so needs less circumference to enclose the same area.]

(c) But the arithmetic mean is also longer than the geometric mean (just take the extremes in one line as diameter of a circle, and compare the perpendicular radius to the perpendicular from the point separating the original extremes).

(d) Kepler seems to be taking it as self-evident that the circumference of the ellipse is greater than that of the circle on J as diameter, and less than that of the circle on N as diameter. Hence it will be close to being the arithmetic mean of these two circles' circumferences. More than that, it is *less* than the circumference of the circle equal to it, i.e. the one on √NJ as diameter. Now, since the circumferences of circles are as their diameters, and √NJ is the geometric mean between N and J, therefore the circumference of the circle on √NJ is also a geometric mean between that of the other two circles. And this geometric-mean-circumference is LESS than the circumference of the ellipse. Therefore the circumference of the ellipse is MORE than the geometric mean between the circumferences of the circles. But so is the arithmetic mean. So that will be close.

Q.E.D.

PROTHEOREM 6



Gnomons of squares divided proportionally are as the squares.

Given: ML : LK = BH : HE

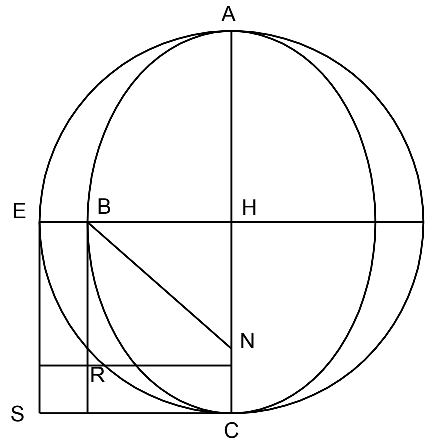
Prove: gnom KOQ : gnom ERC = LP : HS

	$ML^2 : LK^2 = BH^2 : HE^2$	[from the given]
or	LO : LP = HR : HS	[renaming the squares]
so	LP – LO : LP = HS – HR : HS	
i.e.	gnom KOQ : LP = gnom ERC : HS	
alt.	gnom KOQ : gnom ERC : LP : HS	

Q.E.D.

NOTE: The converse is also true.

PROTHEOREM 7



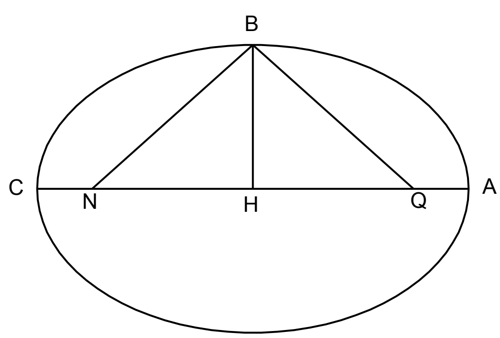
Given: Ellipse center H, major AC, minor HB
Circle center H, radius EH coincident with minor HB
BN = HC (so N is a focus!)
gnomon ERC formed in square EHC, with square RS = EB^2

Prove: HN^2 = gnomon ERC

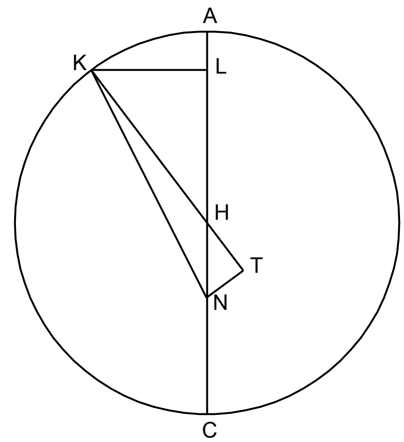
$HN^2 = BN^2 - BH^2$ [1.47]
$HN^2 = HC^2 - BH^2$ [BN = HC, given construction]
HN^2 = gnomon ERC

Q.E.D.

NOTE: If we make HQ = HN, it follows that BQ = NB. But BN = HC by construction, and therefore BN + BQ = 2CH = CA, the major. And since N and Q are equidistant from the center of the ellipse, it follows that N and Q are foci.



"PROTHEOREM 7½" (This is my own addition)



Given:	Circle H, radius HK
	KL perpendicular to the diameter AC
	N any point on HC
	NT perpendicular to HK
Prove:	KT > NL

(This is for the sake of proving that Kepler's orbit-point construction will actually produce a point on KL!)

	NH < HC	[part < whole]
	NH < HK	[HC = HK, radii]
but	NT : KL = NH : HK	[△HTN similar to △HLK]
so	NT < KL	
so	$NT^2 < KL^2$	
so	$KN^2 - NT^2 > KN^2 - KL^2$	[the same, minus the lesser, is greater]
i.e.	$KT^2 > NL^2$	[1.47]
so	KT > NL	

Hence a circle of radius KT, drawn around N as center, will cut perpendicular KL.

But it is furthermore obvious that NK > KT (NK is the hypotenuse in △NKT), and therefore the circle of radius KT, drawn about N as center, must cut KL inside the circle.

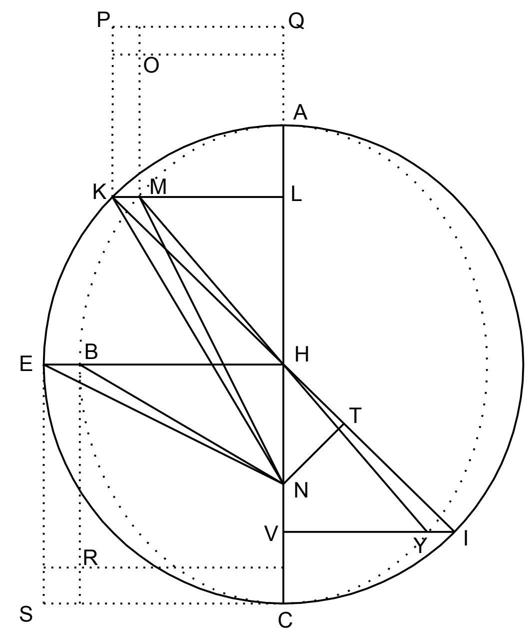
Q.E.D.

KEPLER

DAY 49

THE ORBIT OF MARS IS AN ELLIPSE WITH THE SUN AT ONE FOCUS

Chapter 59
Protheorems XI, XV



PROTHEOREM 11

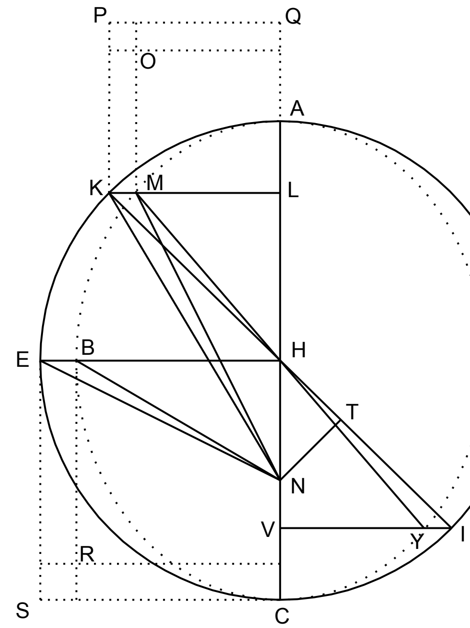
For this proof, Kepler trots out his epicycle again. So first let's do it his way.

In the opening line of the chapter, he refers to this as "the demonstration," i.e. the culmination of the preceding lemmas. He is showing that the orbit of Mars is an ellipse, and the Sun resides at the focus, although his style is so obscure one could easily read the whole chapter and not realize that.

He begins with an ellipse whose major axis AC is the line of apsides in the circle of the good-old-circular hypothesis. He takes equal arcs on that circle, but this is not really crucial for this first demonstration; any arcs will do (though he will, in Protheorem XV, become concerned with comparing arcs on the circle to arcs on the ellipse for the sake of determining whether these are equivalent enough to save the area-clock rule for the ellipse, since he originally developed it for the circle). But he draws HBE perpendicular to AC, and cutting the ellipse and circle in B, E. He picks another point L (it can be random) along AC, and draws LMK perpendicular to AC, cutting the ellipse and circle in M, K. Using EB, the "breadth of the lunule," he completes the gnomon ERC within the square EHCS, and using B as center, HC as radius, he draws a circle, cutting AC at N. Hence BN = HC (as in the construction for Protheorem 7).

He joins KH and extends this through to the circle at *I*, dropping *IV* perpendicular to AC, and cutting the ellipse at Y. He joins NK, NM, NI, NY.

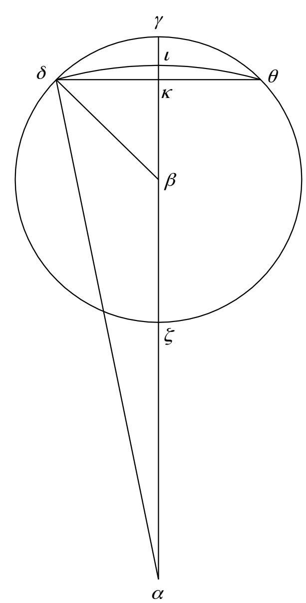
Then he reintroduces the epicycle: “Further, let the diagram of Chs. 39 and 57 be brought back . . .”



Let the Sun be at α, (corresponding to point N)
Let the radius of the epicycle equal the eccentricity, i.e. $\beta\gamma = HN$
Let arc γδ be similar to arc AK
Let αβ be equal to HA, the radius of the eccentric.

Now, Kepler has a two-part enunciation:

(1) $NK = \alpha\delta$, i.e. NK is “the circumferential distance.” But this is a mere equivalence proof, which was done in Ch. 2. To see it quickly here: If we make K the top of the radius of the epicycle, keeping that radius always parallel to itself as the epicycle goes around the deferent, then of course $NK = \alpha\delta$, i.e. $NK = \alpha\theta$, since



	$\alpha\beta = HK$	(by construction)
and	$\beta\theta = HN$	(by construction)
and	$\angle\gamma\beta\theta = \angle AHK$	(by construction)
so	$\angle\theta\beta\alpha = \angle KHN$	(their supplements)
so	$\triangle KHN$ is congruent to $\triangle\theta\beta\alpha$	
so	$NK = \alpha\theta$	
or	$NK = \alpha\delta$	

(2) $NM = \alpha\kappa$, i.e. NM is “the diametral distance,” i.e. the distance determined along the diameter of the epicycle. In other words, the distance to the ellipse is the same as the distance determined by the epicycle which he showed, in Chs. 56 and 57, to be the correct distances to the Martian orbit from point N.

The proof of this second part is as follows.

First	$KN^2 = KL^2 + LN^2$	[1.47]
and	$MN^2 = ML^2 + LN^2$	[1.47]
so	$KN^2 - MN^2 = KL^2 - ML^2$	

Now complete KL^2 and its gnomon on KM, i.e. square KPQL and gnomon KOQ.

So $KL^2 - ML^2 =$ gnomon KOQ
thus $KN^2 - MN^2 =$ gnomon KOQ [***]

Now $KL : EH = KM : EB$ [Protheorem 1]
so $KL^2 : EH^2 =$ gnomon KOQ : gnomon ERC [Protheorem 6]
and $KL : EH = \delta\kappa : \beta\gamma$

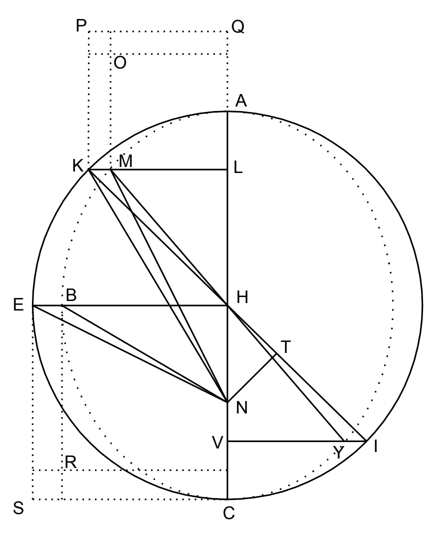
(since KL : EH is the sine of ∠KHL, i.e. of ∠KHA, which, by construction, is equal to ∠δβγ, and therefore sin ∠KHA = sin ∠ δβγ, i.e. KL : EH = δκ : βγ)

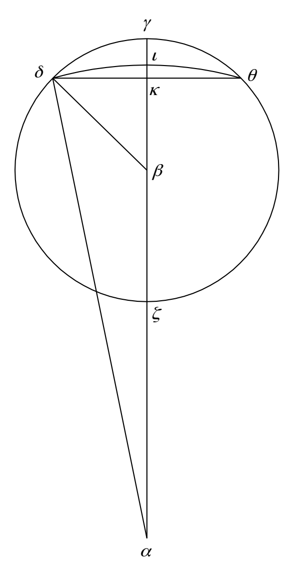
Thus $\delta\kappa^2 : \beta\gamma^2 =$ gnomon KOQ : gnomon ERC

But $\beta\gamma^2 = HN^2$ [epicyclic radius = eccentricity]
and $HN^2 =$ gnomon ERC [by Protheorem 7]
so $\beta\gamma^2 =$ gnomon ERC
so $\delta\kappa^2 =$ gnomon KOQ [looking back to the last proportion]

Thus $\delta\alpha^2 - \kappa\alpha^2 =$ gnomon KOQ [1.47]
i.e. $KN^2 - \kappa\alpha^2 =$ gnomon KOQ [NK = αδ by (1) above]
but $KN^2 - MN^2 =$ gnomon KOQ [by the earlier step, marked *** above]
so $MN = \kappa\alpha$

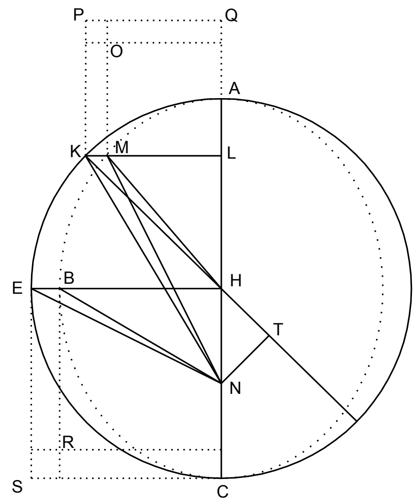
Q.E.D.





Now all of that is more difficult and strange than it needs to be, since we have no more need of an "epicycle," having no special attachment to Kepler's ideas about "reciprocation." Also, he does not bring out, here (as he does in *The Epitome of the Copernican Astronomy*), that N is the focus!

So let's do it again, more simply, *sans epicycle*.



GIVEN: Sun at N
Martian line of apsides ANC, center H
Circle AKEC on diameter AC, center H
KML any perpendicular to AC, cutting Mars's orbit at M
Radius EBH perpendicular to AC, cutting Mars's orbit at B
BN = EH (Ch.56)

PROVE: The orbit of Mars is an ellipse with major axis AC,
and N is a focus.

Join NK, NM, NE, NB.
Join KH, drop NT perpendicular to KH (produced if necessary).

Hence	NM = KT	[Ch. 57, 58]

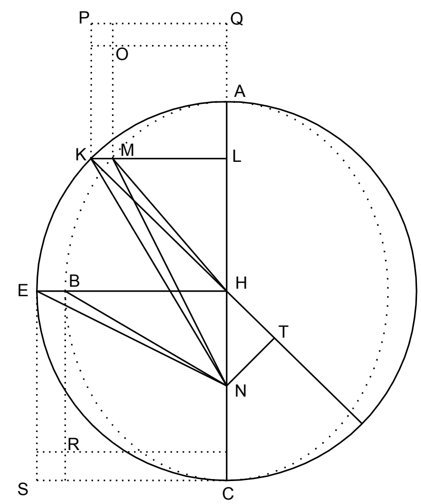
Complete the square on KL (i.e. KPQL) and its gnomon on KM (i.e. gnomon KOQ).
Complete the square on EH (i.e. EHCS) and its gnomon on EB (i.e. gnomon ERC).

Now	$KN^2 - NM^2 = (KL^2 + LN^2) - (ML^2 + LN^2)$	[1.47]
so	$KN^2 - NM^2 = KL^2 - ML^2$	
so	$KN^2 - KT^2$ = gnomon KOQ	[NM = KT]
i.e.	TN^2 = gnomon KOQ	[1.47]
so	TN^2 : gnom ERC = gnom KOQ : gnom ERC	
so	$TN^2 : HN^2$ = gnom KOQ : gnom ERC	[since BN = EH, Protheorem 7 applies]
But	TN : HN = KL : KH	[△KHL and △NHT similar]
so	$KL^2 : KH^2$ = gnom KOQ : gnom ERC	
so	KL : EH = KM : EB	[converse of Protheorem 6]
alt	KL : KM = EH : EB	
so	KL : KL – KM = EH : EH – EB	
so	KL : ML = EH : BH	

But M is a random point on the orbit of Mars. Therefore the orbit of Mars is the locus of points cutting perpendiculars (such as KL) dropped from the circle to its diameter all in the ratio of EH : BH. But the locus of such points is an ellipse with its major axis being the diameter of the circle (converse of Protheorem 1). Therefore the locus of points M is an ellipse with major axis AC, i.e. the orbit of Mars is such an ellipse. And since B is also a point on the ellipse, and HB is perpendicular to the midpoint of major axis AC, HB is the semi-minor axis, and since BN = EH = HC, therefore N is the focus. So the Sun is at one focus of the elliptical orbit of Mars.

Q.E.D.

PROTHEOREM 15



This one begins "But let us complete the proof." What is left to complete?

There appear to be two things left which Kepler wants to establish in this protheorem:

(1) That the proportionality of areas and times in the ellipse, and in the Old Circular Hypothesis, match each other and match the observations.

(2) That his physical hypotheses are now confirmed.

(1) As for the areas and times business, he distinguishes a MAJOR premise, a MINOR, and a CONCLUSION:

MAJOR: Elliptical area AMN is as the sum of the distances from N to elliptical arc AM, and that sum of distances is the same as the sum of the same number of distances from N to the equal parts of the circular arc AK. This he takes as previously demonstrated.

MINOR: Circular AKN : Elliptical AMN = Semicircle AKC : Semi-ellipse AMC.

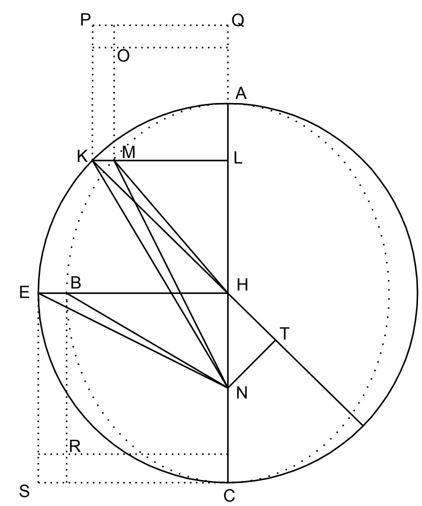
Moreover, if we take arcs AK, AM very small (e.g. 1 degree, as he will do later in the empirical version of this argument), and take a bunch of tiny, equal arcs on the circle AM, etc., then, since "the ratio of equimultiples is the same," he derives the following...

CONCLUSION: Circular area AKN also measures the sum of diametral (or elliptical) distances along arc AM, there being as many of these distances as there are equal arcs on AK (an infinity, if you want to get exact!).

He seems to be aware that he is being unclear, so he goes back to the way he discovered this fact, i.e. empirically. He divides the circle into 360 equal arcs, i.e. individual degrees. If we drew distances from N to the endpoints of these arcs, giving us 360 distances to the circle, then in terms of the radius HC = 100,000 the average distance will be 100,000, so that if we multiply this average times 360 we should get the total, and hence the total of all the unequal distances is 36,000,000. But that is to the circle. What about to the ellipse? For that, we need to replace distance NK with "diametral" distance NM = KT, and distance NI with "diametral" distance NY = TI. He does this for each of his 360 arcs. Now note that KT + TI = 200,000. So too every pair of "diametral distances" = 200,000, and we will have 180 pairs of these, and hence our total distance-sum is still (200,000) × 180 = 36,000,000.

The whole interest in the distance-sums is that he thinks these must be proportional to the times, as we saw before. But he also saw that the areas in the circle were proportional to the elapsed times. Now he needs to see that this will work also for the areas of the ellipse; and he is saying that he discovered the rule of proportions (i.e. that the area AKN : area AMN = semicircle AKC : semi-ellipse AMC) "empirically." This is strange, since that is a matter of pure geometry and, as he says above, was shown in Protheorem 3. Anyway, he bumped into it empirically, or first verified it empirically, it seems.

So he verifies that the proportion holds good, taking the distance-sum in one sector, like AMN, and finding its ratio to the whole 36,000,000 units of distance, and seeing that this is the same ratio as that which the time in arc AM has to the whole period (360° of time). He says that this way of estimating the ratios of areas was quite exact, which is what he means in saying "This produced exactly the same results ... as would have come out had I multiplied half the eccentricity [i.e. ½ HN] by the sine of the eccentric anomaly" [i.e. by the sine of ∠KHA, i.e. by KL]—for this multiplication produces the area of △KHN, which, added to circular sector KHA, and compared to the area of the circle, gives the same ratio.



As often happens, Kepler's zeal for clarity leaves something to be desired. Another way to put it all is thus:

We saw, principally by observation, that areas in our Simple-Circle Hypothesis (in which we initially thought the orbit of Mars was a perfect circle) were as the times in which Mars swept them out around the Sun. That is, when Mars is at M, the time it took to go through arc AM in its orbit is to its whole period as the area NKA is to the area of the whole circle. But we want the area-law to be maintained within the ellipse itself, or to see whether that is the case or not. Well, we already know, by observation, that

(a) Time AM : Whole period of Mars = Area NAK : Whole circle

But, by Protheorem 3, we also know that

(b) Area NAM : Whole ellipse = Area NAK : Whole circle

Hence we now conclude that

(c) Time AM : Whole period of Mars = Area NAM : Whole ellipse

To refresh ourselves on Step (b), reason thus:

	Area AML : Area AKL = ML : KL	[from Prothm. 2]
but	<u>△NML : △NKL = ML : KL</u>	[Euc. 6.1]

so	Area NAM : Area NAK = ML : KL	[adding first ratios]
i.e.	Area NAM : Area NAK = BH : EH	[Prothm. 1, BH : EH = ML : KL]
but	Whole ellipse : Whole circle = BH : EH	[Prothm. 2]
so	Area NAM : Area NAK = Whole ellipse : Whole circle	
alt	Area NAM : Whole ellipse = Area NAK : Whole circle	

Since the areas in the circle are exactly as the times, and the areas in our ellipse are exactly as the areas in the circle, therefore the areas in the ellipse are also exactly as the times. Kepler's Second Law is still safe, despite the transition from circle to ellipse.

(2) PHYSICAL HYPOTHESES CONFIRMED. "And unless the physical causes that I had taken in the place of principles had been good ones, they would never have been able to withstand an investigation of such exactitude," says Kepler.

So he is reasoning **not like this**: "If my physics were right, the following would be true; but the following is true, therefore my physics is right," **but like this:** "If AND ONLY IF my physics were right, then the following would be exactly true, but it is exactly true—everything I check that follows from my physics is exactly true, and therefore my physics is true." This hearkens back to Ch. 21, where he wants to defend his entire method, and say that his way of deriving the truth from his theory proves the truth, not just the probability, of his theory. Much less does his method, as he sees things, achieve no more than agreement with the appearances.

Then again, his magnetic ideas are wrong. His idea about the speeds being inversely as the distances from the sun is not exactly right, either. So how did he get the truth about the shape of the orbit and about the 2nd Law? What "physics" did he get right?

I think we can sum up the correctness of his physics thus:

(a) The motion of the planets is due to the influence of the Sun.

(b) That influence decreases somehow with distance.

(c) Hence the speeds of the planets will decrease with distance from the Sun, and, thanks to the nearly-circular orbits of the planets, the speeds will be almost inversely as the distances.

(d) We might also add that the planets, like earthly stuff, are sluggish, and hence they tend to keep their axes parallel unless something else interferes (contrary to Copernicus's idea). This is more peripheral, however.

(e) The influence of the Sun upon the planets is "immaterial," i.e. passes through intervening empty space and bodies alike.

(f) The influence of the Sun upon the planets is somehow in proportion to the masses of the Sun and each planet (at least he thinks something like this is true between the earth and moon).

(g) The influence of the Sun upon the planets is something like magnetism.

There is a whole lot of physical truth in all that!

KEPLER

DAY 50

“AFTERMATH”

In this final day, we will cover the following material:

(1) The word “focus.”
(2) The near-circularity of the Martian orbit.
(3) What became of the equant.
(4) A regret over the lines of apsides.
(5) A note on the telescope.
(6) What we learn, in the end, about the process of discovery, by reading the *New Astronomy*.

(1) THE WORD “FOCUS”

Apollonius called the foci of a conic section the “points of application,” since he derived them by applying a certain area to a line. It is Kepler who first dubbed such a point a “focus,” from the Latin for “hearth” or “fireplace,” since the Sun sits at one such point in the elliptical orbit of each planet. He does this in *The Epitome of the Copernican Astronomy*, 5.3: “I am accustomed to calling these two points the ‘foci’.” The “focus” or hearth is both a place where the fire is (like the Sun) and it is also the center of the home (as the Sun is the center of the world for Kepler).

(2) THE NEAR-CIRCULARITY OF THE MARTIAN ORBIT

To appreciate Keplerian precision, one must see, visually, just how nearly circular the orbit of Mars is. Not only that, but among the planets that Kepler was studying, Mars was practically the most elliptical, the least circular (and that is not accidental; it is the planet most of all refusing to move in a circle, most obviously “off,” and so most easy to crack, as it were; Mercury is perhaps more elliptical still, but it presents difficulties, as its elliptical orbit has a fairly rapid rate of precession, and it is so close to the sun as to be hard to observe, and it is so close to the sun that there is a complex “wiggle” in its line of apsides. And Pluto—well, he didn’t know about Pluto, and it is no longer ranked among planets).

First of all, here are some numbers:

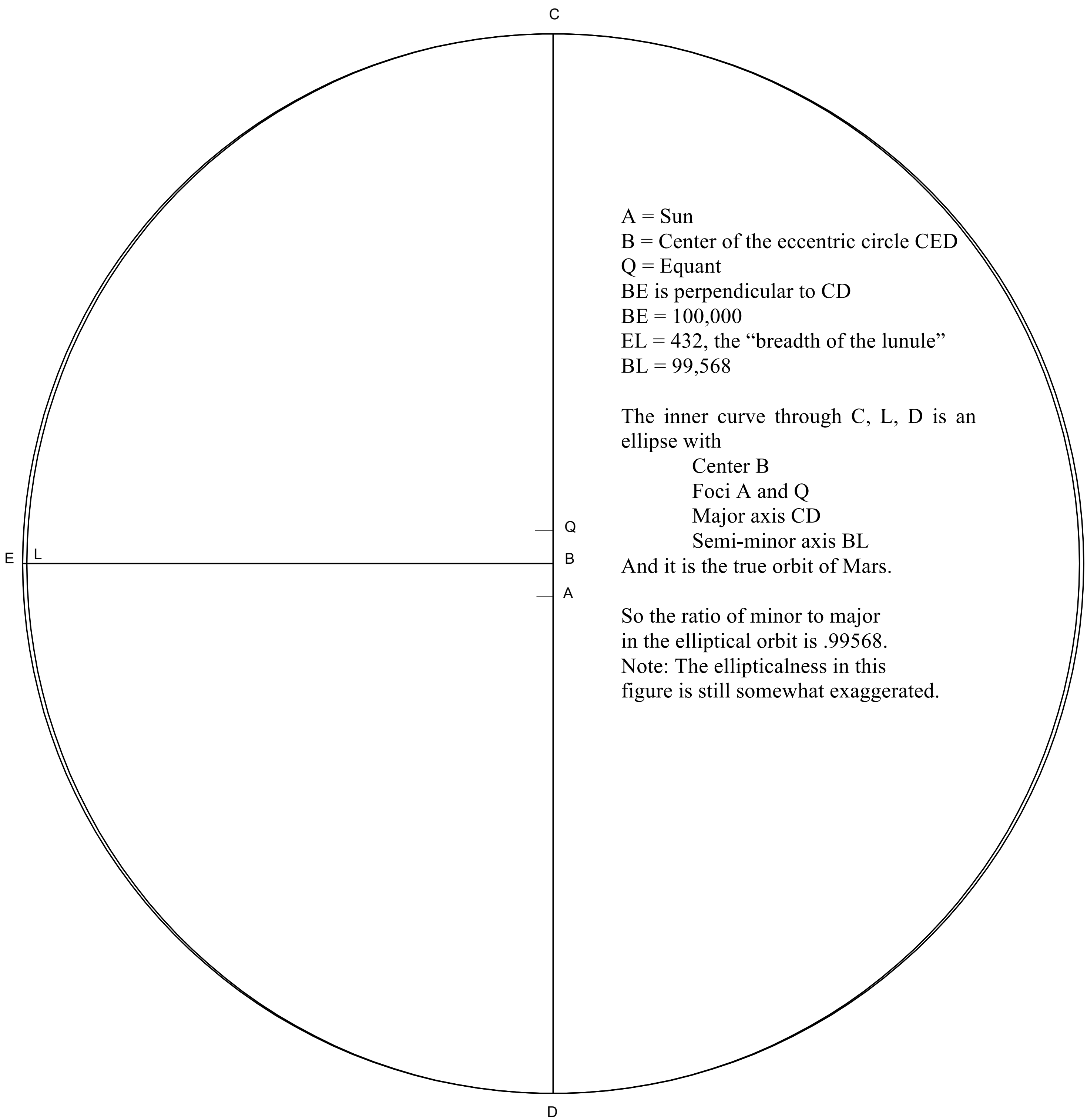
Let BL be the semi-minor, AP the major (and line of apsides), CA the aphelial distance, AD the perihelial distance, BL : BD the ratio of eccentricity or "ellipticalness," and we have, in modern figures, for each of the planets, the following:

	AC	AD	CD	BD	BL	BL : BD
Mercury	47	31	78	39	38.170669	.978735
Venus	73	72	145	72.5	72.498275	.999976218
Earth	101	98	199	99.5	99.4886928	.99988636
Mars	166	138	304	152	151.35	.995749276
Jupiter	545	495	1040	520	519.39869	.998843636
Saturn	1001	900	1901	950.5	949.157521	.998587607
Uranus	2008	1828	3836	1918	1915.8873	.9988985
Neptune	3033	2979	6012	3006	3005.8787	.9999597
Pluto	4930	2960	7890	3945	3820.0524	.9683276

Now, taking that ratio of the major to minor axis, BD : BL (i.e. the halves), and putting them in order of increasing ellipticalness, we get:

Venus	= .999976218	(least elliptical)
Neptune	= .999597	
Earth	= .99988636	
Uranus	= .9988985	
Jupiter	= .998843636	
Saturn	= .998587607	
Mars	= .995749276	
Mercury	= .978735	
Pluto	= .9683276	(most elliptical)

Here is a figure to show just how elliptical the most elliptical orbit is, i.e. the orbit of Mars.



A = Sun
B = Center of the eccentric circle CED
Q = Equant
BE is perpendicular to CD
BE = 100,000
EL = 432, the “breadth of the lunule”
BL = 99,568

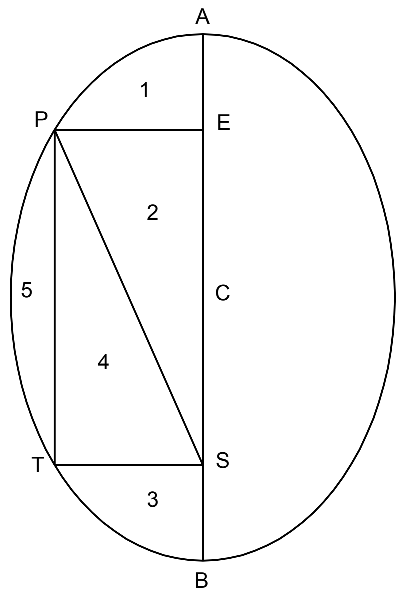
The inner curve through C, L, D is an ellipse with

Center B
Foci A and Q
Major axis CD
Semi-minor axis BL

And it is the true orbit of Mars.

So the ratio of minor to major in the elliptical orbit is .99568.
Note: The ellipticalness in this figure is still somewhat exaggerated.

(3) WHAT BECAME OF THE EQUANT



With the new area-law in place for clocking the motions of planets around the sun, what becomes of the equant? Is it the other focus in the ellipse? Is it the case that the planet sweeps out equal angles in equal times around that other focus?

No. For if possible, let it be so. Let AB be the major axis, S the sun at one focus, E the other focus, and draw EP and ST perpendicular to the axis AB, thus forming equal ordinates. Join SP, ST, PT.

Since E is the presumed equant-point, and $\angle AEP = \angle PEB$, therefore the planet spends equal times in arcs AP, PB.

Since the planet spends equal times in those arcs, in describing them it sweeps out equal areas around the Sun. Hence area SAP = area SPB.

i.e. $1 + 2 = 3 + 4 + 5$

But $1 = 3$
and $2 = 4$ from the symmetry of the ellipse.

Thus $3 + 4 = 3 + 4 + 5$,

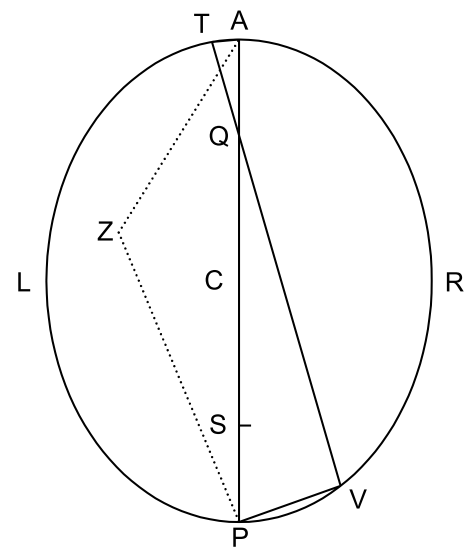
which is absurd.

Therefore, if the area law is true, E cannot be the equant.

Can *any* point be the equant-point? It turns out to be *mathematically impossible!* (How amazing that Kepler discovered the area-law through an equant-point which, strictly speaking, is not compatible with the area-law.) There is a proof for this which is not elementary, so it will not be presented here. But it is mathematically demonstrable that if a body moving on an ellipse sweeps out equal areas in equal times around the focus, it does not sweep out equal angles in equal times around any point at all. The following is an attempt to argue for this in a somewhat elementary fashion.

If a body moves in an ellipse and sweeps out equal areas in equal times around one focus, then there is no point around which it sweeps out equal angles in equal times.

Let a body move on an ellipse with major axis AP, sweeping out equal areas in equal times around focus S.



I say there is no point around which the body sweeps out equal angles in equal times.

For, supposing there were such a point inside the ellipse, it would have to lie along AP. To see this, consider any point Z inside the ellipse but off AP. The angle AZP facing arc PRA will not be equal to the angle AZP facing arc ALP, but the times through arc ALP and arc PRA must be equal, since the areas they embrace are equal. Hence the body sweeps out unequal angles in equal times around Z. Hence Z is not the point around which the body sweeps out equal angles in equal times. Therefore such a point, if there is one, must lie on AP.

Now let the other focus be called Q. Draw ∠AQT very acute, and extend TQ to V (so ∠AQT = ∠VQP). What is the ratio of the times in arc AT and arc PV as this angle about Q shrinks to nothing?

Well, since speed = distance/time, therefore time = distance/speed.

So the time in arc AT is the distance (i.e. arc AT) divided by the average speed in arc AT. Or, as arc AT gets smaller (as $\triangle$t goes to zero),

$$\textit{time in arcAT} = \frac{AT}{\textit{speed at } A} \qquad \text{[ultimately]}$$

since the average speed in arc AT is ultimately equal to the speed at A, and the length of arc AT is ultimately equal to chord AT.

So $$\frac{\textit{time in angle } AQT}{\textit{time in angle } QPV} = \frac{AT/\textit{speed at } A}{PV/\textit{speed at } P}$$

i.e. $$\frac{\textit{time in angle } AQT}{\textit{time in angle } QPV} = \frac{AT}{PV} \cdot \frac{\textit{speed at } P}{\textit{speed at } A}$$

so $$\frac{\textit{time in angle } AQT}{\textit{time in angle } QPV} = \frac{AT}{PV} \cdot \frac{SA}{SP}$$

since, when the areas are as the times around S, then

speed at P : speed at A = SA : SP

as Kepler and Newton both say.

But	TQ : QP = AT : PV	[since △TQA is similar to △PQV]
and	TQ = QA	[ultimately]
so	QA : QP = AT : PV	[ultimately]
but	QA = SP	
and	QP = SA	
so	SP : SA = AT : PV	[ultimately]

thus $$\frac{\textit{time in angle } AQT}{\textit{time in angle } QPV} = \frac{SP}{SA} \cdot \frac{SA}{SP}$$

i.e. time in ∠AQT = time in ∠QPV [ultimately]

So, toward aphelion and perihelion, the times spent in arcs that make equal angles around Q get as close to being equal as we please (as ∠PQT shrinks). Plainly, at any *other* point along AP, the ultimate ratio of such times will *not* be equality, and therefore if there is a point around which the body sweeps out equal angles in equal times *it must be Q*, the empty focus.

Draw QB perpendicular to AP.
Draw SN perpendicular to AP.
Join BN.
Join BS.

By symmetry it is evident that BQSN is a rectangle, and area 1 = area 4, and $\triangle 2 = \triangle 3$.

Now suppose the body sweeps out equal angles in equal times around Q.
Since $\angle AQB = \angle BQP$, the time in arc AB is equal to the time in arc BP. But therefore, by our givens,

Area SAB = Area SBP

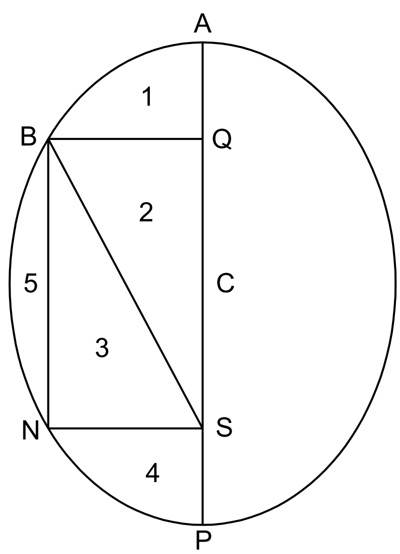
i.e. 1 + 2 = 3 + 4 + 5

which is absurd, since then area 5 would be nothing.

Therefore the body does *not* sweep out equal angles in equal times around Q.

But we saw that if it did sweep out equal angles in equal times around any point at all, it would have to be Q. Therefore it does not sweep out equal angles in equal times around any point at all.

Q.E.D.



(4) A REGRET OVER THE LINES OF APSIDES

In his Introduction to the New Astronomy, Kepler promised to prove that all the lines of apsides of the planets pass through the physical body of the Sun. What a compelling and exciting fact! In the interests of brevity, we mentioned this but did not present it. It is a strong piece of evidence in favor of heliocentrism.

But Kepler's three Laws of Planetary Motion are themselves very convincing evidence, too. The planets all make almost perfectly exact ellipses around the Sun, with the Sun almost exactly at the common focus of all those ellipses—and the same is true for Earth, if we attribute the relative motion between it and the Sun to it and not to the Sun. But if we take Earth as the immobile center, and refer the motions of the planets to it, they certainly do not make exact ellipses around it, with the Earth at the common focus of all. Hm. Almost makes one want to be a heliocentrist, doesn't it? Again, the planets sweep out areas exactly as the times around the Sun—but around Earth, they sometimes stop and stand still, go backwards, start going forward again. It is entirely impossible for them to sweep out areas as times around the Earth. Almost makes it seem as though the Sun, not the Earth, is the significant point around which the planets are moving, doesn't it?

(5) A NOTE ON THE TELESCOPE

Kepler, a contemporary of Galileo, lived to see the invention of the telescope, and to use it. He observed Jupiter's moons, for instance. He lived from 1571-1630.

(6) WHAT WE LEARN, IN THE END, ABOUT THE DISCOVERY PROCESS, BY READING THE *NEW ASTRONOMY*

Kepler's *New Astronomy* is prolix and difficult, and in no small part this is due to its nature as a record of discovery rather than as a clean presentation of facts and demonstrations. Kepler excuses himself for this choice in the Summary to Ch. 45, saying that he finds the discovery process through which the human mind must go every bit as marvelous as the heavenly bodies themselves. So he records his own process, in case we can make something of it, and learn how to make discoveries ourselves. This is directly the opposite of what Euclid does, who hands us demonstrations in perfect clarity, but leaves us utterly in the dark as to how anyone might have discovered such things.

So: What do we learn about discovery from Kepler? Here are a few thoughts, in no particular order.

(1) IMAGINATION is crucial. We have to be able to invent, imagine new ideas quite before testing them. And they must be somehow based on what we already knew before, but not

merely as demonstrable conclusions. We have to be able to follow the hints and suggestions of nature. There is creativity, here. It is not random guesswork, but the efforts of a disciplined imagination.

And we see this in Kepler when he thinks up a new possible path for Mars to follow, or a new rule for determining true distances. And we see even in his writing style that he is endowed with a mighty imagination.

(2) OPENNESS of mind is crucial. If one will not accept anything but the things one is perfectly sure about already, one will discover nothing. One must be willing to entertain possibilities, to range them all, or all the likely ones, in one's mind, without insisting upon judging them, without deciding among them, right away. This means considering even conflicting ideas at the same time, without deciding between them.

We see this when Kepler gives fast-and-loose "proofs" of things, as though that is good enough when we are in discovery mode, i.e. it proves the idea is worthy of more exact consideration.

Along with this, EXPECT THE UNEXPECTED, because you will not be struck by it otherwise. And it is the thing most worth knowing.

(3) ATTENTIVENESS TO STRIKING THINGS is crucial. When we discover, in a way, we are only partly active, but we are largely passive. We actively bring things before our minds, but then we have to sit back and let them strike us, do things to us, get our attention.

We see this in Kepler with his Secant of the Greatest Optical Equation, back in Chapter 56 of the New Astronomy.

(4) THE HUMAN MIND GOES BACK AND FORTH. When people argue, they are arguing "back and forth," on opposite sides of a question. And when we search for the truth, which is what reason spends most of its time doing, it looks back and forth between the alternatives. It reasons backwards and forwards to try to get at the same thing from opposite ends (e.g. analytic geometry). Like the human eyes shifting back and forth, this is the movement of UNCERTAINTY and also of SEARCHING, of attempting to determine a preference between alternatives—it is also the movement of WIDENING one's view to be most likely to take in something significant on the horizon—*scanning*. One might say that the human mind "reciprocates," to use Kepler's own language.

It is typical also of the human mind to change its mind about things. To go back to a former opinion. We see this often in Kepler.

And it is also typical of the human mind to improve old thoughts, even without overturning them, but by refining them by our many revisitations to them. "Defining," for example, is largely a matter of "refining" our initial understanding of what something is. And we see this with Kepler, too. The ellipse itself is not so much a trashing of the initial circular hypothesis, but an adjustment of it, especially in Kepler's view of things.

It is also typical of the human mind to know things better in light of their opposites. So we look back and forth, each time seeing this thing in light of a fresh and recent appreciation of its opposite.

(5) WILLINGNESS TO ABANDON OLD IDEAS is crucial, too. Kepler is quite willing to say "the physical ideas of Ch. 45 go up in smoke." He pours out sweat, blood, tears over

some result, but when he finds it is false, he dumps it without hesitation, and manfully starts again. Such a man *deserves* to discover great things!

On the other hand, KEEPING OLD IDEAS is to some extent crucial. This is a matter of moderation. Hence we keep the initial circular hypothesis at least as a tool, because it gets the equations right (i.e. the times). And we keep the equant, because we believe in a uniformity of planetary motion (whether of divine or natural origin). And if we throw out everything, we no longer have anything from which to start. So Kepler is fond of his "6 physical axioms of very great certainty." We must often come back to our touchstones—to what *we really know*. Mere willingness to doubt everything will get us nowhere.

Also, we see in Kepler a tendency to retain old ideas picked up along the path of discovery in his final idea, even if the old ideas are not really essential to it. Kepler's equant, the circular path, the epicycle, are all instances. He still sees the ellipse as "an adjusted circle," and his physics is meant to cause that adjustment.

(6) GOOD OLD-FASHIONED STUBBORN PERSISTENCE is crucial. If Kepler were willing to fail only once, or only a hundred times, he would not have gotten where he got. "He who would find gold must be willing to dig up much earth and find very little," as one of the pre-Socratics said.

This means we must love the truth and desire to know it strongly enough to motivate us through all that pain and toil. So where wonder is feeble, important intellectual discovery will not take place.

We must even delight in being proved wrong, as a sign that we are a little closer to the truth: we now know that *this* is not it! We have been emancipated from that error! So it has been said that a scientist delights in nothing more than in finding facts that conflict with his theory. This is delightful, too, because our theory is the thing that seemed most plausible to us before—when it turns out false, that means the truth is something stranger than we had thought at first, *and that makes us want to find out what it is all the more*.

(7) HOW THE FALSE LEADS TO THE TRUE is something we see illustrated in the *New Astronomy*, and Kepler even discusses this in Ch. 21. How exactly does the false lead to the true?

(a) INDIRECTLY. What is slightly false can lead to what is entirely true, despite the slight falsehood. For instance, Kepler discovers the area-law based on a slightly false idea of the rules for speeds in a planet. He thinks the speeds are inversely as the solar distances, when really they are inversely as the perpendiculars from the sun to the tangents at the places where we are comparing the speeds.

(b) DIRECTLY, or nearly so. As far as our mode of discovery is concerned, there is something too difficult about discovering the true through the true. The true is exactly what we are trying to find. So what we must do is find the true through the false—that is, through something false which has enough truth in it to bring us to the true, but which *differs from the true* enough that it will occur to us without too much effort.

(c) And this is the importance of NATURAL MISTAKES, errors which are stepping-stones to the discovery of the truth, without which we would not discover the truth. Nor do I think this is entirely a fact of the human mind, but to some extent physical reality itself, as though it were made in order to instruct the human mind, is full of things which will strike our senses, and give us *basically* the right idea, which, upon closer examination, requires further refinement—and then that refinement, too, at the next level of experience of nature, will turn out to need further refinement still, and so on.

(8) WE OFTEN DISCOVER A GENERAL RULE BY FIRST BUMPING INTO THE MOST KNOWN OR SIMPLEST INSTANCE OF IT. This fits with human nature's funny combination of senses and reason. First, in our sense experience, we encounter a particular instance of something, and recognize that it is a particular instance of something, but we are not sure of what. We find an example of this in Kepler with his discovery of the rule of switching the "radius for the secant" in Chapter 56 of the New Astronomy. And he gets the general rule wrong at first.

(9) CHANCE AND LUCK. We can't get these entirely out of the process. Otherwise, there would be a simple procedure: Do A, B, C and you will make a world-class discovery. No such *luck*! We find it in Kepler when he says "Quite by chance" he had been considering the secant of the greatest optical equation in Ch. 56.

(10) WE OFTEN DISCOVER BY RECOGNIZING SOMETHING WE ALREADY KNEW IN A NEW GUISE, or unfamiliar context, e.g. the ellipse under the disguise of being the locus of points constructed by the "reciprocation" rule. This is the "forehead slap" ingredient in nearly all great discoveries.

(11) Last, but by no means least, IN PHYSICS WE OFTEN MAKE DISCOVERIES BY APPROACHING FROM MANY VANTAGE POINTS, i.e. by the harmonizing of PHYSICAL and GEOMETRICAL principles, independently thought-up. This is throughout Kepler.

This is probably the most important element of his method of discovery, and his most unique contribution. Get the physics right, and the math will follow. Get the math right, and the physics will suggest itself. With the truth, all things harmonize (as Aristotle said in his *Nicomachean Ethics*).